Loal height functions in the Arakelov theoretic approach to Quadratic Chabauty I

CHABONNTY

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2.7.2026

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Summary of both talks

Mostly joint with Müller and Srinivasan

I explain the "Arakelov theoretic" approach to Quadratic Chabauty.

As usual, one gets a global height function, which is quadratic, giving a "local at p" expression.

The global height decomposes to a sum of local terms. Away from p they are almost always 0 and always take a finite number of possible values.

The local heights at p are Vologodsky integrals.

For $l \neq p$ one can use the "same" functions over $\mathbb{Q}_l$ to compute the local terms at l by using "differentiation with respect to $\log(l)$".

In computing the local terms at l with this method, the main ingredient is the Katz-Litt Theorem.

I will say something about my proof of this Theorem

Similar ideas give the computation of the local terms away from p.

We recover the results of Betts and Dogra on these terms.

We also have an algorithm for computing the local terms, similar to Betts, Duque-Rosero, Hashimoto and Spelier.

p-adic adelic heights, Data

- F a number field.
- X a smooth variety / either F or a completion thereof.
- p a fixed prime.
- Idele class character $\ell = \sum_{\nu} \ell_{\nu} : \mathbb{A}_F^{\times}/F^* \to \mathbb{Q}_p$.
- $\mathcal{L}$ line bundle on X

Note: ν is always non-archimedean valuation normalized so $\nu(\pi_{\nu}) = 1$, π_{ν} a uniformizer.

For $\nu | p$ we have a decomposition $\ell_{\nu} = t_{\nu} \circ \log_{\nu}$

Local p-adic Adelic metrics

Notation: for a line bundle $\mathcal{L}$, $\mathcal{L}^\times$ = total space minus 0-section.

Definition
Let $K = F_\nu$. A (logarithmic) p-adic metric on a line bundle $\mathcal{L}/X/K$ is

- When $\nu | p$ a function $\mathrm{Log}_\mathcal{L} : \mathcal{L}^\times(\overline{K}) \to \overline{K}$ such that

$$\mathrm{Log}_\mathcal{L}(\alpha w) = \log_\nu(\alpha) + \mathrm{Log}_\mathcal{L}(w),\ \alpha \in K^\times, \quad w \in \mathcal{L}_x, \quad x \in X(\overline{K})$$

which is Galois equivariant over K.

- When $\nu \nmid p$ a valuation, i.e., a function $\mathrm{val}_\mathcal{L} : \mathcal{L}^\times(\overline{K}) \to \mathbb{Q}$ s.t. $\mathrm{val}_\mathcal{L}(\alpha w) = \nu(\alpha) + \mathrm{val}_\mathcal{L}(w)$ for $w \in \mathcal{L}_x$.

p-adic Adelic metrics and heights

Definition
A p-adic adelic metric on $\mathcal{L}/X/F$ is a collection of metrics for all ν + "for almost all ν ..." condition.

Definition
The height associated to an adelic metric on a line bundle $\mathcal{L}$ is the function

$$h_{\mathcal{L}} : X(F) \to \mathbb{Q}_p$$

given by

$$h_{\mathcal{L}}(x) = \sum_{\nu | p} t_\nu(\mathrm{Log}_{\mathcal{L}_\nu}(w)) + \sum_{\nu \nmid p} \mathrm{val}_{\mathcal{L}_\nu}(w) \cdot \ell_\nu(\pi_\nu), \quad w \in \mathcal{L}_x$$

which is independent of w by the "product formula".

Curvature

$K = F_\nu$, $\nu | p$, $\mathcal{L}/X/K$.

$\cup : H^1_{\mathrm{dR}}(X/K) \otimes \Omega^1(X) \to H^2_{\mathrm{dR}}(X/K)$.

We consider certain log functions on $\mathcal{L}$ which are at most double Vologodsky integrals.

Theorem (B. p-adic Arakelov theory)
Suppose $ch_1(\mathcal{L})$ is in the image of $\cup$. Then

1. *To every log function $\mathrm{Log}_{\mathcal{L}}$ on $\mathcal{L}$ we can associate canonically its curvature $\alpha = curve(\mathrm{Log}_{\mathcal{L}}) \in H^1_{\mathrm{dR}}(X/K) \otimes \Omega^1(X)$.*
2. *We have $\cup\alpha = ch_1(\mathcal{L})$.*
3. *Conversely, to any α satisfying the above we can associate a $\mathrm{Log}_{\mathcal{L}}$ with curvature α, unique up to adding an integral of $\omega \in \Omega^1(X)$.*

The explicit computation of log functions from curvature is in general hard, but:

1. Curvature behaves nicely w.r.t. pullbacks.
2. Suppose $\dim X = 1$, $\alpha = \sum [\eta_i] \otimes \omega_i$
   Here $[\eta_i]$ = cohomology class of the form of the second kind η. Then

$$\mathrm{Log}_{\mathcal{L}}(s) = \sum \int \left(\omega_i \int \eta_i \right) + \int \gamma$$

   where γ has a known polar part to make $\mathrm{Log}_{\mathcal{L}}(s)$ have just log singularities with appropriate residues at the divisor of s.
   The integrals can be Vologodsky integrals so it's all agnostic to the reduction type.

Abelian varieties

A/F abelian variety (F arbitrary)

Any $\mathcal{L}/A$ can be written $\mathcal{L}^2 = \mathcal{L}^+ \otimes \mathcal{L}^-$ where

1. $\mathcal{L}^-$ is anti-symmetric (also implies $ch_1(\mathcal{L}) = 0$) $\Rightarrow$ it is "linear".
2. $\mathcal{L}^+$ is symmetric $\Rightarrow$ it is "quadratic".

Linear means: For $m : A \times A \to A$,

$$(*) \qquad m^*\mathcal{L} \cong \pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{L}$$

An (adelic) metric on $\mathcal{L}$ is good if (*) is an isometry

This implies that $h_{\mathcal{L}}$ is linear: $h_{\mathcal{L}}(x+y) = h_{\mathcal{L}}(x) + h_{\mathcal{L}}(y)$.

Similar for quadratic, leading to $h_{\mathcal{L}}$ being a quadratic form.

Real valued good metrics can be obtained by an averaging process, similar to Tate's construction of the canonical height.

For $\nu \nmid p$, they are essentially rational valued, and can be imported to $\mathbb{Q}_p$ (but we have a better? construction)

Theorem (B. Müller, Srinivasan)
Let $\mathcal{L}/A/K$, K p-adic, α a curvature form for $\mathcal{L}$.

1. *For $\mathcal{L}$ symmetric there is a unique good log function on $\mathcal{L}$ with curvature α.*
2. *For $\mathcal{L}$ anti-symmetric there is a good log function on $\mathcal{L}$ with curvature α iff $\alpha = 0$. Then any log function is good.*

Corollary
Any $\mathcal{L}/A/F$ supports an adelic metric s.t. $h_{\mathcal{L}}$ is (at most) quadratic.

Arakelov QC

Data:

1. $X/\mathbb{Q}$ (or number field) complete smooth curve, $\iota : X \to J = J_X$, $\iota(x_0) = 0$.
2. $\mathcal{L}/J$, with $\iota^*\mathcal{L}$ trivial, but $ch_1(\mathcal{L}) \neq 0$.
3. $r = g \Rightarrow (J(\mathbb{Q}) \otimes \mathbb{Q}_p)^* = \mathrm{Span}(\int \omega_i,\ \omega_i \in \Omega(X_p))$.

With a good adelic metric on $\mathcal{L}$, $h_{\mathcal{L}}$ quadratic $=$ polynomial S in $\int \omega_i$.

The section 1 of $\iota^*\mathcal{L}$ decomposes $h_{\mathcal{L}} \circ \iota = h_{\iota^*\mathcal{L}}$

$$h_{\mathcal{L}} \circ \iota = \sum_l h_l$$

$$\begin{aligned} x \in X(\mathbb{Q}) &\Rightarrow S\left(\int_{x_0}^{x} \omega_i\right) = \sum_l h_l(x) \\ &\Rightarrow S\left(\int_{x_0}^{x} \omega_i\right) - h_p(x) = \sum_{l \neq p} h_l(x) \in T \text{ finite} \end{aligned}$$

As usual, to make this work we need:

1. A way to compute h_p
2. A way to find all possible values of h_l for bad reduction primes l.

For the first problem

$$\mathrm{Log}_{\iota^*\mathcal{L}}(1) = \sum \int \left(\omega_i \int \eta_i\right) + \int \gamma$$

determined by the pulled back curvature form (note: reduction agnostic)

For the second problem we can reduce to other approaches.

But we can also apply the same formulas with Vologodsky valuations

Vologodsky valuations

K p-adic for clarity (but later p will morph into $l \neq p$)

The Vologodsky integral depends on the "branch parameter" $\log(\pi)$.

Definition
The Vologodsky valuation associated with a log function $\log_{\mathcal{L}}$ is the function $\mathrm{val}_{\mathcal{L}} : \mathcal{L}^{\times}(\overline{K}) \to \overline{K}$,

$$\mathrm{val}_{\mathcal{L}} = \frac{d}{d\log(\pi)} \mathrm{Log}_{\mathcal{L}} \Big|_{\log(\pi)=0} .$$

Vologodsky valuations

Theorem
The Vologodsky valuation is a (K-valued) valuation.

Proof.
$\log(z) = \log(\frac{z}{\pi^{\nu(z)}}) + \nu(z)\log(\pi)$ so $\frac{d}{d\log(\pi)}\log(z) = \nu(z)$. $\square$

$\mathcal{L}/A/K$ l-adic field, Good log function $\Rightarrow$ good Vologodsky valuation (provided it is $\mathbb{Q}$-valued, which it is !!)

For QC, only need the pullback valuation on $\iota^*\mathcal{L}$ $\Leftarrow$ pulled back log function $\Leftarrow$ pulled back curvature form by

$$\mathrm{Log}_{\mathcal{L}}(1) = \sum \int \left(\omega_i \int \eta_i\right) + \int \gamma$$

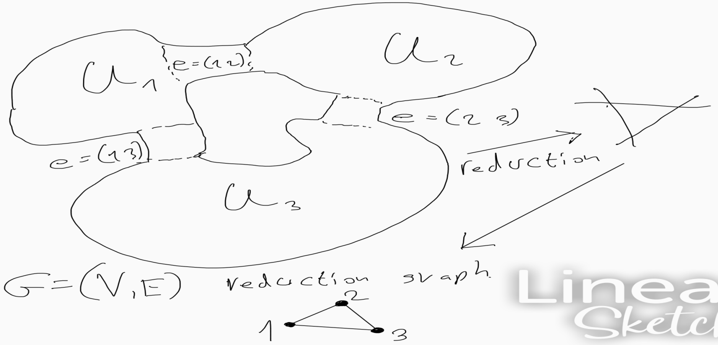
So we need to differentiate w.r.t. $\log(\pi)$.

This can be done in the semi-stable reduction case using the Katz-Litt Theorem.

Explicit computation on curves

Consider

- X/K a complete curve with semi-stable reduction
- $\Gamma = (V, E)$ the reduction graph



Graph convention: Each edge has a start $e+$ and end $e-$ and a reversed edge $-e$ where the order is reversed.

Crash course on graph cohomology

$d : K^V \to K^E_-$, $df(e) = f(e^+) - f(e^-)$.

$H^1(\Gamma, K) = K^E_-/dK^V$.

The dual (w.r.t. obvious scalar products) $d^* : K^E_- \to K^V$, $d^*g(v) = \sum_{e^+=v} g(e)$.

Harmonic cochains $\mathcal{H}(\Gamma, K) = \operatorname{Ker} d^*$.

For a connected Γ the harmonic decomposition $K^E_- = dK^V \oplus \mathcal{H}(\Gamma, K)$ follows by observing that the kernel of the Laplacian $\Delta = d^* \circ d : K^V \to K^V$ is the space of constant functions and its image is it orthogonal complement.

The theorem of B. - Zerbes

ω - a form of the second kind on X. $F = \int \omega$, $F_v = F|_{U_v}$.

$e = (v, w) \mapsto \chi_\omega(e) = F_v - F_w \in K$

Theorem (B., Zerbes)
The F_v are Coleman integrals and χ_ω is harmonic.

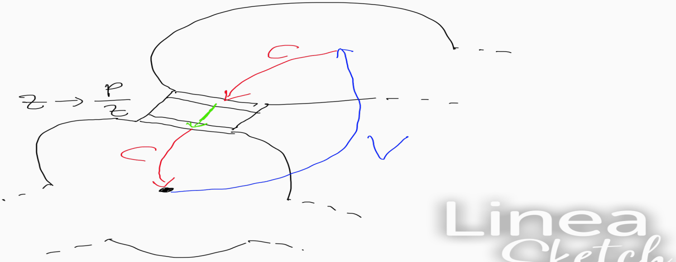
This computes F : Start with any choice of F_v. If χ_ω is not harmonic

1. Decompose $\chi_\omega = c_{har} + df$.
2. adjust $F_v \to F_v - f(v)$

Explicitely: Adjust F_v by $\Delta^{-1} d^* \chi_\omega$.

The Katz-Litt Theorem

Relates Coleman and Vologodsky paths, between the fibers at various points of a unipotent connection (M, ∇).



A loop as above going through an edge e gives rise to a monodromy matrix $A(e)$ (which is always unipotent).

Theorem (Katz-Litt)
The association $e \to A(e)$ is harmonic in the sense that for each $v \in V$, $\sum_{e+=v} \log(A(e)) = 0$. Also, within the same U_v, Vologodsky = Coleman.

Annuli and orientations

Reference: Coleman - Reciprocity laws on curves

An annulus is a rigid space e with

$$z : e \xrightarrow{\sim} \{r < |z| < 1\}.$$

Induced residue map: $\omega \in \Omega^1(e)$

$$\omega = \sum a_i z^i dz.$$

$$\mathrm{res}_z\, \omega = a_{-1}.$$

$$\mathrm{res}_z : H^1_{\mathrm{dR}}(e) \xrightarrow{\sim} K.$$

All res_z are the same up to sign, splitting z's into 2 orientations.

The reverse orientation to that of z is given by $z' = \alpha/z$ with $|\alpha| = r$.

Oriented annulus = annulus e + orientation and has res_e

The ring of log functions

For an oriented annulus e with parameter z define

$$\mathcal{O}_{\log}(e) = \mathcal{O}(e)[\log(z)]\ , \quad \Omega^1_{\log}(e) = \mathcal{O}_{\log}(e)dz.$$

$\log(z)$ is a formal variable and there is a differential
$d : \mathcal{O}_{\log}(e) \to \Omega^1_{\log}(e)$ such that $d\log(z) = dz/z$.

The ring $\mathcal{O}_{\log}(e)$ is independent of the choice of the parameter z within the same orientation up to a canonical isomorphism: If z' is another parameter the two rings are canonically isomorphic using

$$\log(\frac{z}{z'}) \in \mathcal{O}_e$$

For reverse orientations this is almost true: If $z' = \pi/z$ then

$$\log(z') = \log(\pi) - \log(z)$$

so the isomorphism depends on the choice of $\log(\pi)$.

Vologodsky's nearby cycles functor

e - an oriented annulus

(M, ∇) a unipotent connection on e.

$M \cong \mathcal{O}(e)^n$.

Derivation δ on $\mathcal{O}_{\log}(e)$, $\delta(P(\log(z))) = P'(\log(z))$ derivative with respect to $\log(z)$.

d and δ commute.

Definition
The Vologodsky nearby cycle of (M, ∇) is

$$\Psi_e(M, \nabla) := \left(M \otimes_{\mathcal{O}(e)} \mathcal{O}_{\log}(e)\right)^{\nabla=0},$$

By the commutation of d and δ, $W = \Psi_e(M, \nabla)$ is preserved by δ, inducing a nilpotent operator N on W.

$z \to \pi/z$ gives $T : \Psi_e(M, \nabla) \to \Psi_{-e}(M, \nabla)$.

Example

$\omega \in \Omega^1(e)$ with $\mathrm{res}_e\, \omega = r$

$$\omega = \sum a_i z^i dz \ , \quad a_{-1} = r.$$

$M = \mathcal{O}(e)^2$,

$$\nabla(y_1, y_2) = (dy_1, dy_2 - \omega y_1).$$

$$\Psi_e(M, \nabla) = Span((0,1),(1,F)) \ , F = \int \omega.$$

$F = r\log(z) + G$ with $G \in \mathcal{O}(e)$ so $\delta F = r$ and

$$N(0,1) = 0, \quad N(1,F) = r(0,1).$$

$$[N] = \begin{pmatrix} 0 & \mathrm{res}_e\, \omega \\ 0 & 0 \end{pmatrix}.$$

Non-abelian harmonic decomposition

Think of ordinary harmonic decomposition as saying:

For any cocycle $e \mapsto c(e)$ there is a function on vertices $v \mapsto f(v)$ such that $e \mapsto c(e) + f(e^+) - f(e^-)$ is harmonic

The version with a non-abelian group G should say:

For any G-valued cocycles $e \mapsto c(e)$ there is a G-valued function on vertices $v \mapsto f(v)$ such that $e \mapsto f(e^+)c(e)f(e^-)^{-1}$ is harmonic (whatever that means)

If G is unipotent, with Lie algebra $\mathfrak{g}$, we can think of G as $\mathfrak{g}$ with the BCH product $*$. When harmonic is via the logs as before, it now reads:

For any $\mathfrak{g}$-valued cocycles $e \mapsto c(e)$ there is a $\mathfrak{g}$-valued function on vertices $v \mapsto f(v)$ such that $e \mapsto f(e^+) * c(e) * -f(e^-)$ is harmonic (in the usual sense)

This can be solved uniquely "diagonal by diagonal" using commutative harmonic decomposition.

This can be used for computation, but also for the proof. Sketch

1. Use non-abelian harmonic decomposition to show that for a given (M, ∇) the Katz-Litt conditions determine the path uniquely.
2. Show that this path is compatible with morphisms and tensor products, so it is really a Tannakian path.
3. Show that it satisfies the Vologodsky conditions:
    3.1 Show that it is fixed by Frobenius.
    3.2 Show that it is killed by an appropriate power of the monodoromy operator.

It is also used for the computation of the derivative with respect to $\log(\pi)$, which is rather similar to the last step of the proof.