Loal height functions in the Arakelov theoretic approach to Quadratic Chabauty II

CHABONNTY

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Summary of yesterday

Mostly joint with Müller and Srinivasan

I explain the "Arakelov theoretic" approach to Quadratic Chabauty.

The global height decomposes to a sum of local terms.

The local heights at p are Vologodsky integrals.

The local heights for $l \neq p$ can be computed using Vologodsky valuations.

We explained the Katz-Litt Theorem.

Today

Computation of the local terms away from p.

Something about the proof of Katz-Litt

An algorithm for computing the local terms

Vologodsky valuations

The Vologodsky integral depends on the “branch parameter” $\log(\pi)$.

Definition
The Vologodsky valuation associated with a log function $\log_{\mathcal{L}}$ is the function $\mathrm{val}_{\mathcal{L}} : \mathcal{L}^{\times}(\overline{K}) \to \overline{K}$,

$$\mathrm{val}_{\mathcal{L}} = \frac{d}{d\log(\pi)} \mathrm{Log}_{\mathcal{L}} |_{\log(\pi)=0} \ .$$

On a line bundle on a curve, the log function will look like

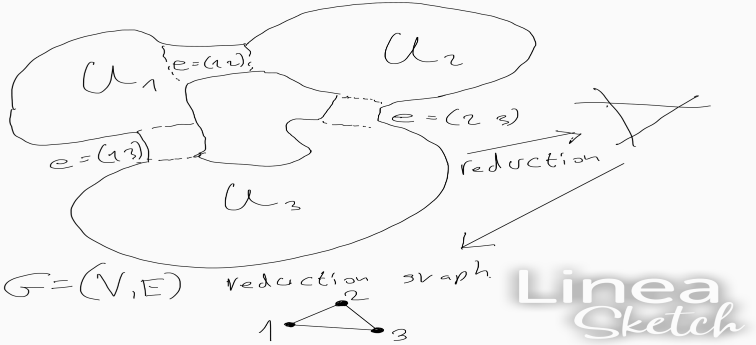
$$\mathrm{Log}_{\mathcal{L}}(m) = \sum \int \left(\omega_i \int \eta_i \right) + \int \gamma$$

So we need to differentiate w.r.t. $\log(\pi)$.

Explicit computation on curves

Consider

- X/K a complete curve with semi-stable reduction
- $\Gamma = (V, E)$ the reduction graph



Graph convention: Each edge has a start $e+$ and end $e-$ and a reversed edge $-e$ where the order is reversed.

$d : K^V \to K_-^E$, $dh(e) = h(e^+) - h(e^-)$.

$H^1(\Gamma, K) = K_-^E / dK^V$.

$d^* g(v) = \sum_{e^+ = v} g(e)$.

Harmonic cochains $\mathcal{H}(\Gamma, K) = \operatorname{Ker} d^*$.

Harmonic decomposition $K_-^E = dK^V \oplus \mathcal{H}(\Gamma, K)$

Laplacian $\Delta = d^* \circ d : K^V \to K^V$

Formula: $f = dh +$ harmonic $\Rightarrow \Delta h = d^* f$.

The theorem of B. - Zerbes

ω - a form of the second kind on X. $F = \int \omega$, $F_v = F|_{U_v}$.

$e = (v, w) \mapsto \chi_\omega(e) = F_v - F_w \in K$

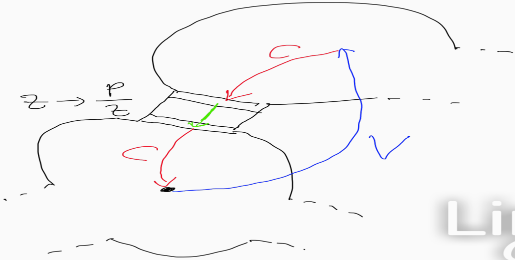
Theorem (B., Zerbes)
The F_v are Coleman integrals and χ_ω is harmonic.

This computes F : Start with any choice of F_v. If χ_ω is not harmonic

1. Decompose $\chi_\omega = c_{har} + dh$.
2. adjust $F_v \to F_v - h(v)$

The Katz-Litt Theorem

Relates Coleman and Vologodsky paths, between the fibers at various points of a unipotent connection (M, ∇).



A loop as above going through an edge e gives rise to a monodromy loop $A(e)$ (which is always unipotent).

Theorem (Katz-Litt)
The association $e \to A(e)$ is harmonic in the sense that for each $v \in V$, $\sum_{e+=v} \log(A(e)) = 0$. Also, within the same U_v, Vologodsky = Coleman.

Annuli and orientations

An annulus is a rigid space e with

$$z : e \xrightarrow{\sim} \{r < |z| < 1\}.$$

Induced residue map: $\omega \in \Omega^1(e)$

$$\omega = \sum a_i z^i dz.$$

$$\mathrm{res}_z\, \omega = a_{-1}.$$

$$\mathrm{res}_z : H^1_{\mathrm{dR}}(e) \xrightarrow{\sim} K.$$

All res_z are the same up to sign, splitting z's into 2 orientations.

The reverse orientation to that of z is given by $z' = \alpha/z$ with $|\alpha| = r$.

Oriented annulus = annulus e + orientation and has res_e

The ring of log functions

For an oriented annulus e with parameter z define

$$\mathcal{O}_{\log}(e) = \mathcal{O}(e)[\log(z)] \ , \quad \Omega^1_{\log}(e) = \mathcal{O}_{\log}(e)dz.$$

$\log(z)$ is a formal variable and there is a differential
$d : \mathcal{O}_{\log}(e) \to \Omega^1_{\log}(e)$ such that $d\log(z) = dz/z$.

The ring $\mathcal{O}_{\log}(e)$ is independent of the choice of the parameter z within the same orientation up to a canonical isomorphism: If z' is another parameter the two rings are canonically isomorphic using

$$\log(\frac{z}{z'}) \in \mathcal{O}_e$$

For reverse orientations this is almost true: If $z' = \pi/z$ then

$$\log(z') = \log(\pi) - \log(z)$$

so the isomorphism depends on the choice of $\log(\pi)$.

Vologodsky's nearby cycles functor

e - an oriented annulus

(M, ∇) a unipotent connection on e.

$M \cong \mathcal{O}(e)^n$.

Derivation δ on $\mathcal{O}_{\log}(e)$, $\delta(P(\log(z))) = P'(\log(z))$ derivative with respect to $\log(z)$.

d and δ commute.

Definition
The Vologodsky nearby cycle of (M, ∇) is

$$\Psi_e(M, \nabla) := \left(M \otimes_{\mathcal{O}(e)} \mathcal{O}_{\log}(e)\right)^{\nabla=0},$$

By the commutation of d and δ, $W = \Psi_e(M, \nabla)$ is preserved by δ, inducing a nilpotent operator N on W.

$z \to \pi/z$ gives $T : \Psi_e(M, \nabla) \to \Psi_{-e}(M, \nabla)$.

Example

$\omega \in \Omega^1(e)$ with $\operatorname{res}_e \omega = r$

$$\omega = \sum a_i z^i dz \ , \quad a_{-1} = r.$$

$M = \mathcal{O}(e)^2$,

$$\nabla(y_1, y_2) = (dy_1, dy_2 - \omega y_1).$$

$$\Psi_e(M, \nabla) = Span((0,1),(1,F)) \ , F = \int \omega.$$

$F = r\log(z) + G$ with $G \in \mathcal{O}(e)$ so $\delta F = r$ and

$$N(0,1) = 0, \quad N(1,F) = r(0,1).$$

$$[N] = \begin{pmatrix} 0 & \operatorname{res}_e \omega \\ 0 & 0 \end{pmatrix}.$$

Dependency of transition on $\log(\pi)$

The map T depends on the choice of the branch parameter $s = \log(\pi)$ as follows

$$T_s = T_0 \exp(sN)$$

In the example above this will correspond to the following behavior in the B.-Zerbes Theorem:

If

1. The annulus $e = U_v \cap U_w$
2. ω is a meromorphic form on X
3. $r = \operatorname{res}_e \omega$
4. F_v and F_w are Coleman integrals of ω on U_v and U_w respectively, for the branch with $\log(\pi) = 0$, so they have a logarithmic term $r \log(z)$
5. $F_v - F_w = c$

Then for a different branch $c \to c + r \log(\pi)$

$\int \omega$ independent of $\log(\pi)$

This is to be expected, at least for holomorphic ω.

Suppose ω is of the second kind.

$F = \int \omega$ Vologodsky integral, for the branch $\log(\pi) = 0$.

$e \mapsto \chi_\omega(e)$ - associated harmonic cocycle.

If we change $\log(\pi)$, $\chi_\omega(e) \to \chi_\omega(e) + \log(\pi)\,\mathrm{res}_e(\omega)$.

So, in principle we may need to adjust F_v to make χ harmonic again

But the residue theorem implies that $e \mapsto \mathrm{res}_e(\omega)$ is harmonic !!

So F_v do not change.

We need to do the same for rank 3.

$$dy_0 = 0, \quad dy_1 = \eta y_0, \quad dy_2 = \omega y_1$$

a basis of solutions:

$$\upsilon^0 = (0,0,1), \quad \upsilon^1 = (0,1,F), \quad \upsilon^2 = (1,G,H) \tag{1}$$

$F = \int \omega$, $G = \int \eta$, $H = \int (G \cdot \omega)$.

Notation:

$$[a,b,c] = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} .$$

Note:

$$X * Y = X + Y + \frac{1}{2}[X,Y]$$

and

$$[[a_1,b_1,c_1],[a_2,b_2,c_2]] = [0,0,a_1 b_2 - a_2 b_1]$$

Fix $\log(\pi) = 0$. Turn operators to matrices by fixing a basis and Vologodsky (with this branch) and Coleman transporting to all points.

We get at every point the basis (1), with F, G, H Vologodsky integrals.

$$[A_e] = \exp([c_\omega(e), c_\eta(e), d(e)])$$

($d(e)$ irrelevant).

$$[N_e] = [\mathrm{res}_e(\omega), \mathrm{res}_e(\eta), \langle G, F \rangle_e]$$

$\langle G, F \rangle_e$ is the double index at e.

What changes if we change $\log(\pi) = s$?

$$\begin{aligned}\log(A_e) &\to \log(A_e) * s[N_e] \\ &= [\chi_\omega(e), \chi_\eta(e), d(e)] * s[\mathrm{res}_e(\omega), \mathrm{res}_e(\eta), \langle G, F\rangle_e] \\ &= [\chi_\omega(e) + s\,\mathrm{res}_e(\omega), \chi_\eta(e) + s\,\mathrm{res}_e(\eta), d(e) + sf(e)]\end{aligned}$$

with

$$f(e) = \frac{1}{2}\left(\chi_\omega(e)\,\mathrm{res}_e\,\eta - \chi_\eta(e)\,\mathrm{res}_e\,\omega\right) + \langle G, F\rangle_e\ .$$

Now we need to fix this by non-abelian harmonic decomposition.

Not surprisingly, the first diagonal is OK.

The top right corner is fixed by $v \mapsto [0, 0, s \cdot h(v)]$ with

$$\Delta h = d^* f$$

THE LINEARITY IN s IS THE ESSENCE OF THE PROOF OF KATZ-LITT !!

Theorem
Suppose ω and η are two forms of the second kind on X. Let

$$H = \int (\omega \int \eta)$$

Then

$$\frac{d}{d \log(\pi)} H = h$$

and

$$\Delta h(v) = \sum_{e+=v} \frac{1}{2}(\chi_\omega(e)\,\mathrm{res}_\eta(e) - \mathrm{res}_\omega(e)\chi_\eta(e)) + \langle \int \eta, \int \omega \rangle_e$$

To compute Vologodsky valuations we have to take into account the section m and the fixer form γ.

Definition
The toric divisor for a section m

$$\mathrm{div}_t(m)(v) = \text{ total degree of } (m) \text{ on } U_v$$

Explicit formula for Vologodsky valuations

Once all is accounted for, the formula becomes cohomological.

For two forms ω, η and any vertex v we have a "global index on U_v" $\langle \omega, \eta \rangle_v$

We define $r : \wedge^2 H^1_{\mathrm{dR}}(X/K) \to K^V$

$$r(\omega, \eta)(v) = \langle \eta, \omega \rangle_v + \frac{1}{2} \sum_{e+=v} (\chi_\omega(e) \operatorname{res}_\eta(e) - \operatorname{res}_\omega(e) \chi_\eta(e))$$

Theorem
Let $\mathrm{val}_{\mathcal{L}}$ *be the Vologodsky valuation associated to a log function on a line bundle* $\mathcal{L}$ *whose curvature is* α *and let* m *be a section of* $\mathcal{L}$*. Then* $\mathrm{val}_{\mathcal{L}}(m)$ *is constant* $h(v)$ *on the component* U_v*, except near the divisor of* m*, and one has* $\Delta h = r(\alpha) - \mathrm{div}_t(m)$

Note: For QC we only need $\alpha = ch_1(\mathcal{L}) \in H^2_{\mathrm{dR}}(J/K) = \wedge^2 H^1_{\mathrm{dR}}(X/K)$

Making it computable - what we need to know

Note the cup product formula

$$\omega \cup \eta = \sum_v r(\omega, \eta)(v) = \sum_v \langle \eta, \omega \rangle_v + \sum_e (\chi_\omega(e)\,\mathrm{res}_\eta(e) - \mathrm{res}_\omega(e)\chi_\eta(e))$$

H^1 of a curve with semi-stable reduction

Weight decomposition $H = H^1_{\mathrm{dR}}(X/K) = H_0 \oplus H_1 \oplus H_2$ where H_i is where Frobenius acts with weight i. Set $H_{ij} = H_i \oplus H_j$.

Facts

- $H_0 \cong H^1(\Gamma, K)$.
- The map $H \xrightarrow{\mathrm{res}} \mathcal{H}(\Gamma, K) \to H^1(\Gamma, K) \to H_0$ equals the monodromy operator N.
- $\mathrm{Ker}(N) = H_{01}$.
- H_0 is in the kernel of $H^1_{\mathrm{dR}}(X/K) \to \oplus_v H^1_{\mathrm{dR}}(U_v)$.

The algorithm

Assumptions

- We have forms of the second kind ω_i, $i = 1, \ldots, 2g$ whose classes generate $H^1_{\mathrm{dR}}(X/K)$.
- We can compute the residues in the Preparatory steps below.
- We know which annuli bound which domain.

Preparatory steps

Step 1: Compute the residues $\operatorname{res}_e \omega_i$ for all i, e.

Step 2: Compute for any pair ω_i, ω_j, any annulus e and any point x the residue $\operatorname{res}_e F_{\omega_i}\omega_j$ and $\operatorname{res}_x F_{\omega_i}\omega_j$ for a choice of an integral F_{ω_i} of ω_i on e or x.

The rest is linear algebra

Step 3. Compute cup products $\omega_i \cup \omega_j$ by summing residues computed in step 2 over the x's

The algorithm

Step 4: Compute the common kernel of res_e, i.e., $\operatorname{Ker} N = H_{01}$.

Step 5: Compute $(\operatorname{Ker} N)^{\perp} = H_0$, the orthogonal complement of H_{01} w.r.t the cup product.

Step 6: The cohomology class α is in the p-eigenspace of Frobenius. Therefore, in $\wedge^2 H$ it is in $H_0 \wedge H_2 \oplus \wedge^2 H_1 \subset H_0 \wedge H + H_{01} \wedge H_{01}$. Use steps 4+5 and linear algebra to write α as such a sum.

Step 7: Compute r on (the part of α) in $\wedge^2 H_{01}$. For $\omega, \eta \in H_{01} = \operatorname{Ker} N$

$$r(\omega \wedge \eta)(v) = \langle \omega, \eta \rangle_v = \sum_e \operatorname{res}_e F_\omega \cdot \eta + \sum_x \operatorname{res}_x F_\omega \cdot \eta.$$

The sum is over e connected to U_v and $x \in U_v$. The integral F_ω are local on e, x - The constant of integration does not matter since $\omega, \eta \in \operatorname{Ker} N$, so they are linear combinations of the numbers computed in step 2.

The algorithm - Step 8: Compute r on $H_0 \wedge H$

Suppose $\omega \in H_0$, $\eta \in H$.

By facts $\langle \omega, \eta \rangle_v = 0$ for all v.

Also $\operatorname{res}_e \omega = 0$ for all e. So

$$r(\omega \wedge \eta)(v) = \sum_{e+=v} \chi_\omega(e) \operatorname{res}_\eta(e)$$

so it is enough to compute χ_ω for all such ω.

The cup product formula gives

$$\omega \cup \omega_i = \sum_e \chi_\omega(e) \operatorname{res}_e(\omega_i)$$

The $e \mapsto \operatorname{res}_e \omega_i$ generate $\mathcal{H}(\Gamma, K)$ so $\omega \cup \omega_i$ recover χ_ω.