Rational Points on Entanglement Modular Curves

Abbey Bourdon (Wake Forest University)

This work was partially supported by NSF grant DMS-2145270.

Isolated Points

Let C be a "nice" curve defined over a number field k. A closed point $x \in C$ of degree d is **isolated** if it does not belong to an infinite family of degree d points parameterized by a geometric object.

- **$\mathbb{P}^1$-parameterized**

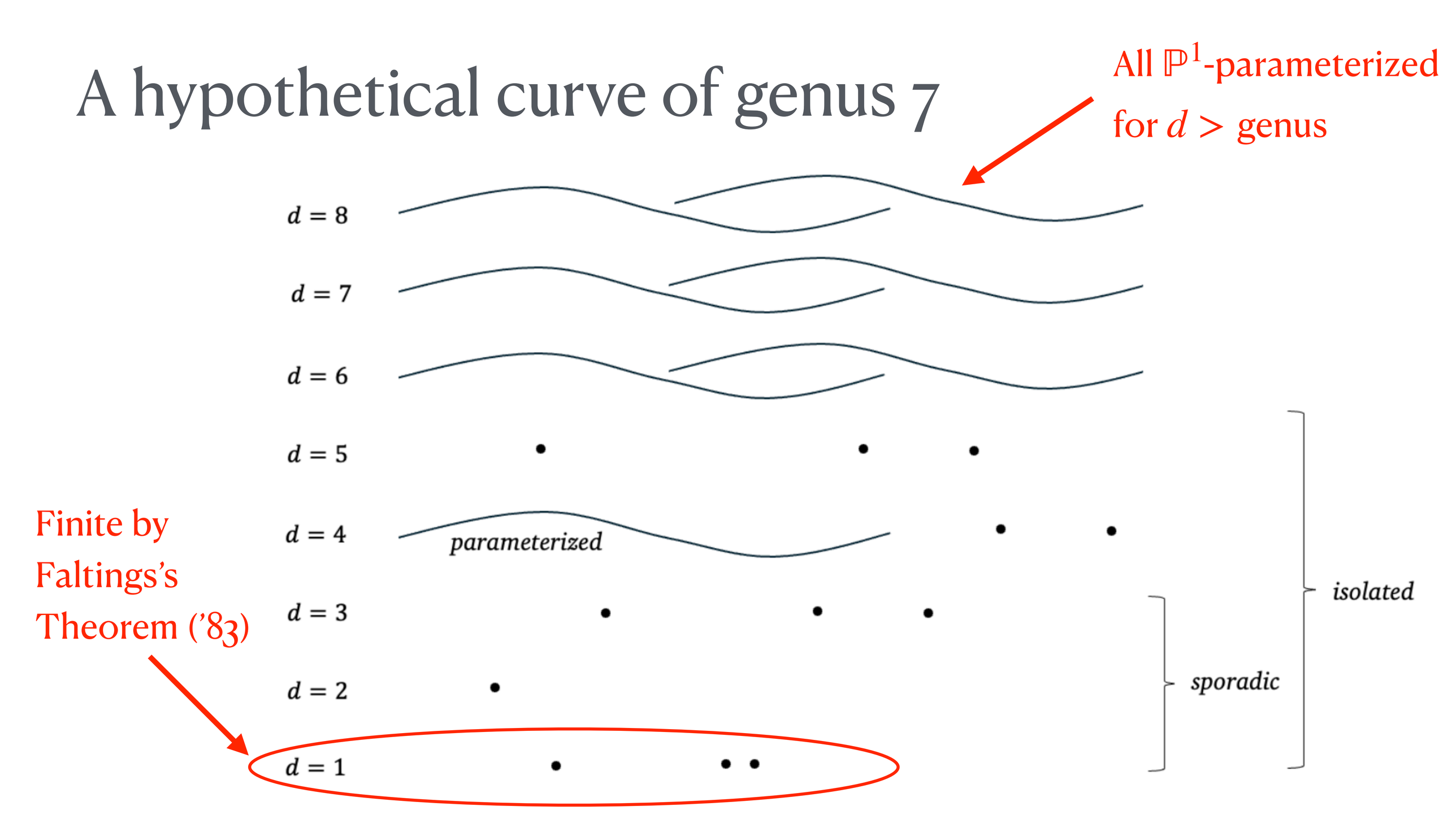
  $f : C \to \mathbb{P}^1$ a rational map of degree d with $x \in f^{-1}(\mathbb{P}^1(\mathbb{Q}))$

- **AV-parameterized**

  $\Phi : C^{(d)} \to \mathrm{Jac}(C)$

  $\Phi(x) + A \subseteq \mathrm{im}(\Phi)$ with A/k a positive rank abelian subvariety of $\mathrm{Jac}(C)$

A hypothetical curve of genus 7



In fact, there are only **finitely many** isolated points on C/k.

B., Ejder, Liu, Odumodu, Viray ('19), as a consequence of Faltings's Theorem ('94)

The Modular Curve $X_1(n)$

The modular curve $X_1(n)$ is a smooth projective curve over $\mathbb{Q}$ whose (non-cuspidal) points parameterize elliptic curves with a point of order n.

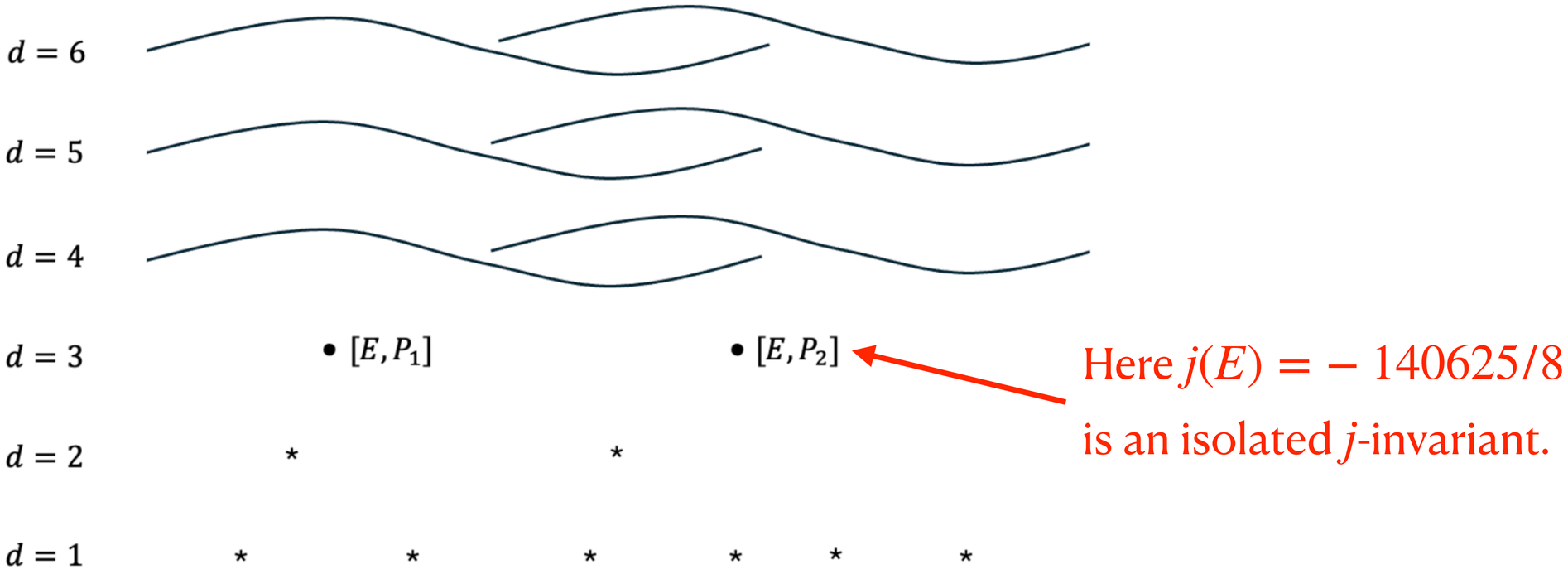
Theorem (Mazur, '77): $X_1(N)(\mathbb{Q})$ has a non-cuspidal point iff $N \leq 12,\ N \neq 11$. For each N, there are infinitely many rational points.

Theorem (Kenku, Momose, '88, Kamienny, '92): $X_1(N)$ has a non-cuspidal quadratic point iff $1 \leq N \leq 18,\ N \neq 17$. For each N, there are infinitely many quadratic points.

Theorem (Derickx, Etropolski, van Hoeij, Morrow, Zureick-Brown, '21): $X_1(N)$ has a non-cuspidal cubic point iff $1 \leq N \leq 21,\ N \neq 17,19$. For each $N \neq 21$, there are infinitely many cubic points.

Isolated j-invariants for $X_1(n)$

We say $j \in X_1(1) \cong \mathbb{P}^1$ is **isolated** if it is the image of an isolated point on the modular curve $X_1(n)$ for some $n \in \mathbb{Z}^+$ under the natural map $j : X_1(n) \to X_1(1)$ sending $[E, P] \to j(E)$.



Degree d points on $X_1(21)$.

Isolated j-invariants for $X_1(n)$

We say $j \in X_1(1) \cong \mathbb{P}^1$ is **isolated** if it is the image of an isolated point on the modular curve $X_1(n)$ for some $n \in \mathbb{Z}^+$ under the natural map $j : X_1(n) \to X_1(1)$ sending $[E, P] \to j(E)$.

Known isolated j-invariants in $\mathbb{Q}$

$-\frac{140625}{8}$: Associated to degree 3 points on $X_1(21)$; Najman ('16).

-9317: Associated to degree 6 points on $X_1(37)$; van Hoeij ('12).

$\frac{351}{4}$: Associated to degree 9 point on $X_1(28)$; B., Gill, Rouse, Watson ('24).

$-\frac{16267752311383867}{7}$: Associated to degree 18 point on $X_1(37)$; Derickx, van Hoeij ('25).

Any of the 13 CM j-invariants in $\mathbb{Q}$: B., Ejder, Liu, Odumodu, Viray ('19).

Theorem: (B., Ejder, Liu, Odumodu, Viray, '19)
Suppose Serre's Uniformity Conjecture holds. Then there are only **finitely many** isolated j-invariants in $\mathbb{Q}$.

It has been conjectured that the 17 j-invariants on the previous slide is the **complete set** of isolated j-invariants in $\mathbb{Q}$. See B., Hashimoto, Keller, Klagsbrun, Lowry-Duda, Morrison, Najman, Shukla ('25) and Terao ('25).

Theorem: (B., Ejder, Liu, Odumodu, Viray, '19)
Suppose Serre's Uniformity Conjecture holds. Then there are only **finitely many** isolated j-invariants in $\mathbb{Q}$.

Unconditional progress:

- Isolated points of odd degree (B., Gill, Rouse, Watson, '24).
- Curves $X_1(p^n)$ for a prime p (B., Ejder, '26).
- Curves $X_1(p^n q^m)$ for distinct primes p, q (B.).

Explicit Results for Non-CM isolated j-invariants in $\mathbb{Q}$

- Isolated points of odd degree (B., Gill, Rouse, Watson, '24).

  $-140625/8,\ 351/4$

- Curves $X_1(p^n)$ for a prime p (B., Ejder, '26).

  $-9317,\ -16267752311383867$

- Curves $X_1(p^n q^m)$ for primes p, q (B.)

$-140625/8,\ 351/4,\ -9317,\ -16267752311383867,\ \ldots ???$

Problem: Can you prove this list is complete? We know it is *finite*.

To move beyond prime-power level, one must control certain *entanglement* of torsion point fields.

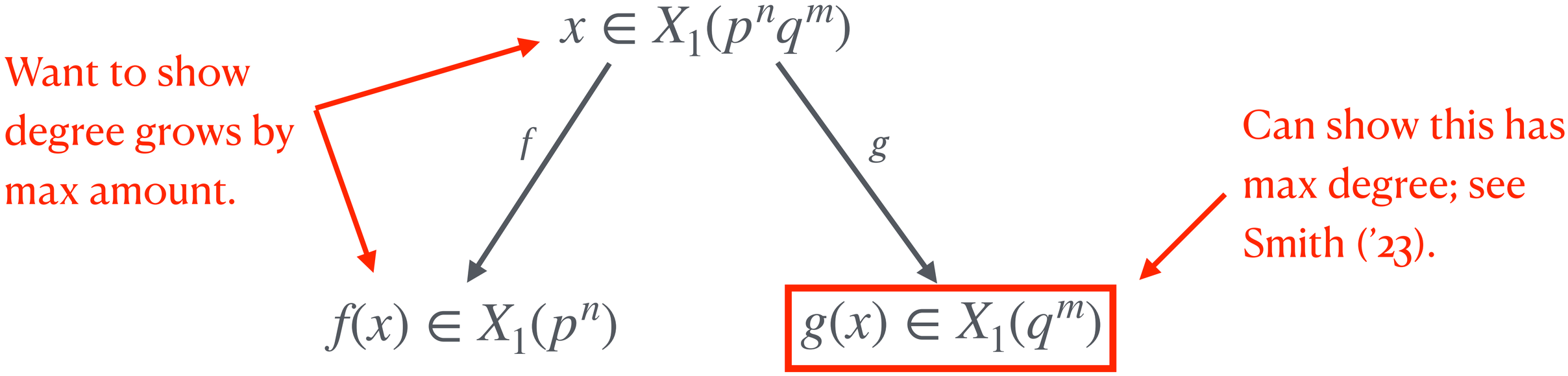
If $\mathbb{Q}(E[p^n]) \cap \mathbb{Q}(E[q^m]) \neq \mathbb{Q}$ for distinct primes p and q, then we say there is **(horizontal) Galois entanglement**. This has been studied in many other works.

- Brau/Jones (‘16)
- Morrow (’19)
- Daniels/Morrow (’22)
- Daniels/Lozano-Robledo (’23)
- Daniels/Lozano-Robledo/Morrow (’23)
- Daniels/Rouse (’26)

*However, only **certain** entanglement of torsion point fields impact the degree of residue fields!*

Isolated Points on $X_1(p^n q^m)$

Let $E/\mathbb{Q}$ be a non-CM elliptic curve and $q > 37$ prime. By work of Mazur ('78), Serre ('81), Bilu, Parent, Rebolledo ('13), and Furio, Lombardo ('23), the mod q Galois representation associated to E is surjective or equal to the normalizer of a nonsplit Cartan subgroup.



Isolated Points on $X_1(p^n q^m)$

Let $E/\mathbb{Q}$ be a non-CM elliptic curve with $x = [E, P] \in X_1(p^n q^m)$ isolated. Suppose $q > 37$.

Case 1: The mod q Galois representation is surjective. There is no entanglement of residue fields (B., Ejder, Liu, Odumodu, Viray, '19).

$$\implies \deg(x) = \deg(f) \cdot \deg(f(x)) \text{ for } f : X_1(p^n q^m) \to X_1(p^n).$$

$$\implies f(x) \in X_1(p^n) \text{ is isolated (B., Ejder, Liu, Odumodu, Viray, '19).}$$

$$\implies j(E) \text{ known (B., Ejder, '26)}$$

Let $E/\mathbb{Q}$ be a non-CM elliptic curve with $x = [E, P] \in X_1(p^n q^m)$ isolated. Suppose $q > 37$.

Case 2: The mod q Galois representation is the normalizer of a nonsplit Cartan subgroup: $C_{ns}^+(q)$. The point $[E, p^n P] \in X_1(q^m)$ has residue field with q highly ramified; see Smith ('23).

$$\implies \text{We have } \deg(x) \geq \frac{1}{6} \cdot \deg(f) \cdot \deg(f(x)) \text{ where}$$

$$f : X_1(p^n q^m) \to X_1(p^n) \text{ is the natural map.}$$

$\implies$Lemos ('19) implies $\deg(f(x))$ is almost always large as possible.

$\implies \deg(x) > \text{genus}(X_1(p^n q^m))$, and x is not isolated.

Example (B., Gill, Rouse, Watson, '24): Show there are no isolated points $x \in X_1(54)$ with $\deg(x)$ odd and $j(x) \in \mathbb{Q}$.

Let $E/\mathbb{Q}$ with $j(x) = j(E)$. Suppose x lies above a cubic point on $X_1(2)$. If entanglement of residue fields occurs, then $\mathbb{Q}(E[2]) \cap \mathbb{Q}(E[27])$ is either a cyclic cubic or S_3 extension. Then E gives a rational point on the modular curve X_H for certain $H \leq \mathrm{GL}_2(\mathbb{Z}/54\mathbb{Z})$. If x is isolated, there are few choices for H, and all rational points on X_H are cusps!

Entanglement that makes residue fields unexpectedly small after compiling across multiple primes typically does *not* occur for CM elliptic curves. **The modular curves characterizing this kind of entanglement are often much easier to analyze.**

Another Example: $X_{ns}^{+}(5) \times X_{ns}^{+}(7)$

Suppose $E/\mathbb{Q}$ has mod 5 image $C_{ns}^{+}(5)$ and mod 7 image $C_{ns}^{+}(7)$.

$\Longrightarrow$ E gives a non-cuspidal rational point on $X_{ns}^{+}(5) \times X_{ns}^{+}(7)$.

$X_{ns}^{+}(5) \times X_{ns}^{+}(7)$: genus 13, analytic rank 13, rational CM points

There are 13 possible mod 35 images for $E/\mathbb{Q}$. All but 3 would give a single closed point on $X_1(35)$ of maximal degree. Exceptions:

35.420.29.f.1, 35.840.59.q.1, 35.840.59.r.1

The last two cover 35.420.29.f.1, which admits a map to a rank 0 elliptic curve. No rational points by Mayle, Rouse ('26).

Why does entanglement **impacting the degree of residue fields** make the corresponding modular curves easier to analyze?

Thank you!