

A Chabauty-Coleman bound for Surfaces

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Part I:

The method of Chabauty and Coleman for curves

Rational points on curves of genus $g \geq 2$

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- **Chabauty's theorem '41:** If $\text{rk } J(\mathbb{Q}) < g$, then $C(\mathbb{Q})$ is finite.
- **Faltings's theorem '83:** Mordell's conjecture is true.

Sketch of Chabauty's proof

Take $x_0 \in C(\mathbb{Q})$, if any. Embed C into J via $x \mapsto [x - x_0]$. Let $r = \text{rk } J(\mathbb{Q})$.

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- The theory of p -adic Lie groups implies that $\dim \Gamma \leq r$.
- Hence, $\dim \Gamma < g = \dim J$.
- $C(\mathbb{Q}_p)$ generates $J(\mathbb{Q}_p)$, so it is not contained in Γ .
- It follows that $C(\mathbb{Q}_p) \cap \Gamma$ is finite.
- Finally, note that $C(\mathbb{Q}) = C(\mathbb{Q}_p) \cap J(\mathbb{Q}) \subset C(\mathbb{Q}_p) \cap \Gamma$.

Chabauty-Coleman bound

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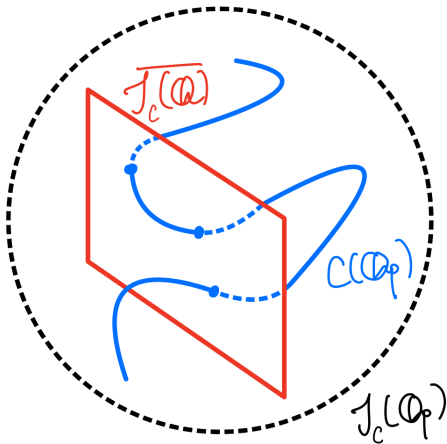
Theorem (Coleman 1985)

Let $C/\mathbb{Q}$ be a smooth projective curve of genus $g \geq 2$ and p a prime of good reduction for C . We assume:

- $p > 2g$,
- Dimension inequality:
 $\text{rk } J(\mathbb{Q}) + 1 \leq g$,

then

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + 2g - 2.$$



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$$X = \underbrace{C + C + \cdots + C}_{d \text{ times}} \quad (\text{Essentially } \mathrm{Sym}^d C).$$

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- **Klassen '93**: Finiteness on a p-adic open set.
- **Siksek '09**: Over number fields. Practical procedure for computations.
- **Park '16**: A conditional bound (not of Coleman type) under certain technical assumptions from tropical geometry.

Part II:

A Chabauty–Coleman bound for surfaces

Results over $\mathbb{Q}$ - also available over number fields

Notation: For a smooth variety V with good reduction at p , the reduction is V' .

Main Theorem (C-, Pasten '23)

Let X be a nice surface of general type in an abelian variety A of dimension 3 over $\mathbb{Q}$, and let p be a prime of good reduction for X and A .

We assume:

- $p > (\frac{128}{9}) \cdot (c_1^2(X))^2$,
- $\text{rk } A(\mathbb{Q}) \leq 1$ (Dimension inequality),
- X' contains no elliptic curves over $\mathbb{F}_p^{alg}$ "finiteness".

Then

$$\begin{aligned} \#X(\mathbb{Q}) &\leq \#X'(\mathbb{F}_p) + \frac{p-1}{p-2} \cdot (p + 4p^{1/2} + 5) \cdot c_1^2(X) \\ &< \#X'(\mathbb{F}_p) + 4p \cdot c_1^2(X) \end{aligned}$$

Recall: $c_1^2(X)$: Self-intersection of the canonical divisor K_X of X .

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Then

$$\#X(\mathbb{Q}) < \#X'(\mathbb{F}_p) + 4p \cdot c_1^2(X)$$

Remark: Compare to Coleman's bound for curves:

$$\#C(\mathbb{Q}) \leq \#C'(\mathbb{F}_p) + c_1(C).$$

Remark

- Plenty of examples when $\text{End}(A_{\mathbb{C}}) = \mathbb{Z}$, using a theorem of Chavdarov on primes of geometrically simple reduction. This works for instance on nice surfaces contained in the Jacobian of $y^2 = x^7 - x - 1$.

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- Plenty of examples when $\text{End}(A_{\mathbb{C}}) = \mathbb{Z}$, using a theorem of Chavdarov on primes of geometrically simple reduction. This works for instance on nice surfaces contained in the Jacobian of $y^2 = x^7 - x - 1$.
- **Main Theorem** is also available when $\dim A > 3$, but we need the existence of an ample divisor that satisfies a certain condition that depends on p , and an extra condition on A' .

Idea of the proof

Setup

- $\Gamma = \overline{A(\mathbb{Q})}$ is a p -adic analytic 1-parameter subgroup of $A(\mathbb{Q}_p)$. Note:

$$X(\mathbb{Q}) = X(\mathbb{Q}_p) \cap A(\mathbb{Q}) \subset X(\mathbb{Q}_p) \cap \Gamma.$$

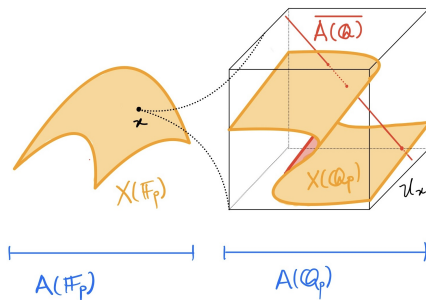
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- **Reduction map:** $\text{red}: A(\mathbb{Q}_p) \rightarrow A'(\mathbb{F}_p)$. For each residue disk $U_x = \text{red}^{-1}(x)$ with $x \in X'(\mathbb{F}_p)$ we want to bound $\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x$.



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- Let f be a local equation for X on U_x . Then $f \circ \gamma(z)$ is a p -adic power series and

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- Write $f \circ \gamma(z) = \sum_n a_n z^n \in \mathbb{Q}_p[[z]]$.
- To bound $n_0(f \circ \gamma(z), 1/p)$ we have to find a small N such that $|a_N| \geq 1$.

This last requirement is **main difficulty** in the whole approach.

Large coefficient in low degree

Definition (ω -integral)

Let k be a field. Let Z, Y be k -schemes and $\omega \in H^0(Y, \Omega_Y^1)$. A k -morphism $\phi : Z \rightarrow Y$ is ω -integral if the composition

$$\phi^\bullet : H^0(Y, \Omega_Y^1) \rightarrow H^0(Y, \phi_* \phi^* \Omega_Y^1) = H^0(Z, \phi^* \Omega_Y^1) \rightarrow H^0(Z, \Omega_Z^1).$$

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Notation: Let $k := \mathbb{F}_p^{alg}$ and $V_m := \text{Spec } k[z]/(z^{m+1})$.

Suppose that there is $m < p$, such that $f \circ \gamma$ satisfies $|a_i| < 1$ for every $i \leq m$.

Then, there exists a closed immersion

$$\phi_m : V_m \rightarrow X_k,$$

at x which is w_i -integral for some $w_1, w_2 \in H^0(X_k, \Omega_{X_k/k}^1)$ independent.

Large coefficient in low degree: the overdetermined method

- We need to bound the m such that the following holds:
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- **The “overdetermined” bound:** We bound m in terms of the geometry of $D = \text{div}(w_1 \wedge w_2)$.

Geometric bounds for ω -integral curves

Let S be a smooth surface over k , let $x \in S$, let $w_1, w_2 \in H^0(S, \Omega_{S/k}^1)$ be independent over $\mathcal{O}_S$ and let $D := \operatorname{div}(w_1 \wedge w_2) = \sum_{j=1}^q a_j D_j$ with D_j irreducible curves and let $\nu_j : \widetilde{D}_j \rightarrow D_j \subseteq S$ be the normalizations.

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Lemma (the bound on m)

Let $\phi : V_m \rightarrow S$ be a closed immersion supported at x . If ϕ is w_i -integral for both $i = 1, 2$ then

$$m \leq \sum_{j=1}^q \sum_{y \in \nu_j^{-1}(x)} a_j (\operatorname{ord}_y(\nu_j^\bullet w_i) + 1).$$

Bounds on U_x

A p -adic analysis argument implies that

$$\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq n_0(f \circ \gamma(z), 1/p) \leq 1 + \frac{p-1}{p-2} \cdot m(x).$$

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- Assume $x \notin \text{supp}(D)$. We have $m(x) = 0$. Then

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- Assume $x \in \text{supp}(D)$. **We use:** the Riemann hypothesis for singular curves, intersection theory computations, controlling singularities of D , etc.

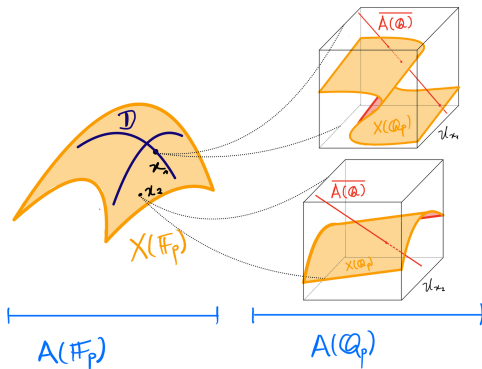
When we sum over all of these points, we get a bound in terms of the prime p and $c_1^2(X) = (D.D)$.

Last step

$$\begin{aligned}\#X(\mathbb{Q}_p) \cap \Gamma &= \sum_{x \in X'(\mathbb{F}_p)} \#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq \sum_{x \in X'(\mathbb{F}_p)} \left(1 + \frac{p-1}{p-2} \cdot m(x) \right) \\ &< \#X'(\mathbb{F}_p) + \frac{p-1}{p-2} \cdot \underbrace{(p + 4p^{1/2} + 5) \cdot c_1^2(X)}_{\text{terms } m(x) \text{ for } x \in D(\mathbb{F}_p)}.\end{aligned}$$

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 \end{aligned}$$



Part III:

A refined Chabauty–Coleman bound for
surfaces

Algebraic points on curves

Let C be a nice curve of genus $g \geq 3$. For $0 < d < g$, we define

$C^{(d)}$ d -th symmetric power of C .

$C^{(d)}(\mathbb{Q})$ parametrizes degree s -points for $s \leq d$.

We have $\phi: C^{(d)} \rightarrow J_C$ and we define $W_d := \text{Im}(\phi)$.

Example ($d = 2$)

When C is not hyperelliptic, ϕ is an isomorphism. Then

$$W_2(\mathbb{Q}) = \{\text{unordered pairs of rational points \& quadratic points}\}$$

Application: quadratic points

A consequence of Main Theorem is:

Corollary (C-, Pasten)

Let $C/\mathbb{Q}$ be a nice non-hyperelliptic curve of genus $g = 3$ whose Jacobian J satisfies $\text{rk } J(\mathbb{Q}) \leq 1$. Let $p \geq 521$ be a prime of good reduction for C . Suppose that C' is not hyperelliptic and that $(C')^{(2)}$ does not contain elliptic curves over $\mathbb{F}_p^{\text{alg}}$. Then $C^{(2)}(\mathbb{Q}) = W_2(\mathbb{Q})$ is finite and

$$\begin{aligned} \#W_2(\mathbb{Q}) &\leq \#W'_2(\mathbb{F}_p) + 6 \cdot \frac{p-1}{p-2} \cdot (p + 4p^{1/2} + 5) \\ &< \#W'_2(\mathbb{F}_p) + 7.2 \cdot p. \end{aligned}$$

Hyperelliptic case

Example

If C is a nice hyperelliptic curve of genus $g \geq 3$, $C^{(2)}$ is the blow-up of W_2 at the origin. The exceptional divisor is

$$\nabla := \overline{\{[(x, y), (x, -y)] : (x, y) \in C\}}.$$

Then

$$W_2(\mathbb{Q}) = \{\text{unordered pairs of rat. pts. \& quadratic pts. lying outside } \nabla\}$$

Theorem (Balakrishnan, C- '25)

Let C be a hyperelliptic curve/ $\mathbb{Q}$ of genus $g = 3$ whose Jacobian has rank ≤ 1 . Let $p \geq 11$ be a prime of good reduction for C . Suppose that W'_2 does not contain elliptic curves over $\mathbb{F}_p^{\text{alg}}$. Then:

$$\#W_2(\mathbb{Q}) \leq \#W'_2(\mathbb{F}_p) + 2p + 12\sqrt{p} + 7.$$

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We implemented an algorithm based on our method with H. Pasten.

Input: hyperelliptic curve C .

Output: An upper bound for the number of rational points of W_2 .

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Remark

In ongoing joint work with S. Müller and R. Parankimamvila, we prove an analogous result for the symmetric square of Picard curves. In this setting, the symmetric square $C^{(2)}$ is naturally isomorphic to W_2 .

Example

Let C be the curve defined by

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We prove $W_2(\mathbb{Q}) = \{0_J\} \cup \{P_i : i = 1, \dots, 6\}$, where

$$P_1 = (0, 0) + 0_J,$$

$$P_2 = (Q) + (Q^\sigma),$$

$$P_3 = (2\sqrt{-1}, 0) + (-2\sqrt{-1}, 0),$$

$$P_4 = (-2 + 2\sqrt{2}, 0) + (-2 - 2\sqrt{2}, 0),$$

$$P_5 = (2 + \sqrt{7}, 0) + (2 - \sqrt{7}, 0),$$

$$P_6 = \iota P_2,$$

$$Q = \left(\frac{-13+2\sqrt{-14}}{9}, \frac{12560-7045\sqrt{-14}}{2187} \right) \text{ and } Q^\sigma \text{ is the Galois conjugate of } Q.$$

Notice that for $p = 5$ and $z = \overline{(0,0)} + \overline{(1,0)} \in W_2(\mathbb{F}_5)$, we have that

$$\{P_2, P_6, (2\sqrt{-1}, 0) + (0, 0)\} \subset W_2(\mathbb{Q}_5) \cap \overline{J(\mathbb{Q})} \cap U_z.$$

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The annihilator of $J(\mathbb{Q})$ under the integration pairing is spanned by

$$\begin{aligned}\omega_1 &= (4 + 5 + 4 \cdot 5^3 + O(5^4)) \frac{dx}{y} + (3 + 2 \cdot 5 + O(5^4)) \frac{xdx}{y}, \\ \omega_2 &= (4 + 4 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + O(5^4)) \frac{dx}{y} + (3 + 2 \cdot 5 + O(5^4)) \frac{x^2 dx}{y}.\end{aligned}$$

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So, we have

$$(w_1 \wedge w_2) = (4x_1x_2 + 2(x_1 + x_2) + 3) \frac{d(x_1x_2) \wedge d(x_1 + x_2)}{y_1y_2}.$$

Then $D = Z_{W_2}(4x_1x_2 + 2(x_1 + x_2) + 3)$. Notice that $z \in D$.

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- For $i = 1, 2$ we have

$$\text{ord}_{z_i}(\nu^\bullet w_1) = 0$$

Then

$$m(z) \leq \sum_{j=1}^q \sum_{y \in \nu^{-1}(x)} a_j(\text{ord}_y(\nu^\bullet w_1) + 1) = 2.$$

Finally, we have

$$\#W_2(\mathbb{Q}_5) \cap \overline{J(\mathbb{Q})} \cap U_z \leq 1 + \frac{4}{3}m(z) = 3.\bar{6}$$

Thank you very much for your attention!