Beyond Quadratic Chabauty

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Including joint work with I. Dan-Cohen and M. Lüdtke
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Faltings–Siegel

Theorem (Faltings-Siegel)

For a smooth irreducible curve $\mathcal{X}$ over $R := \mathcal{O}_k[1/S]$ for a number field k and finite set S of places of k, and for which $\mathcal{X}(\mathbb{C})$ has negative Euler characteristic,

$$|\mathcal{X}(R)| < \infty.$$

More concretely, the following sets are finite:

- $\{x, y \in R^\times \mid x + y = 1\}$ (AKA $\mathcal{X} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$)
- For $a, b \in R$ such that $4a^3 + 27b^2 \in R^\times$,
  $\{x, y \in R \mid y^2 = x^3 + ax + b\}$ (AKA $\mathcal{X}$ a punctured elliptic curve)
- $X(k)$ for a curve X of genus $g \geq 2$ ($\mathcal{X}$ an integral model of X)

Remark

The $\mathcal{X}$ for which $\mathcal{X}(\mathbb{C})$ has negative Euler characteristic are precisely those for which $\pi_1(\mathcal{X}(\mathbb{C}))$ is non-abelian.

Chabauty and Chabauty–Kim

Chabauty's approach to proving cases of Faltings' Theorem:[i]

$$\begin{tikzcd}
\mathcal{X}(R) \arrow[r, "\mathrm{loc}_{\mathcal{X}}"] \arrow[d] & \mathcal{X}(\mathbb{Z}_p) \arrow[d] \arrow[dr] & \\
J(R) \arrow[r, "\mathrm{loc}_J"] & J(\mathbb{Z}_p) \arrow[r, "\log_J"] & \mathrm{Lie}(J_{\mathbb{Q}_p})
\end{tikzcd}$$

Kim replaced this with the diagram:

$$\begin{tikzcd}
\mathcal{X}(R) \arrow[r, "\mathrm{loc}_{\mathcal{X}}"] \arrow[d, "\kappa"] & \mathcal{X}(\mathbb{Z}_p) \arrow[d, "\kappa_p"] \arrow[dr, "\int_{BC}"] & \\
H^1_{f,R}(G_k; U_n) \arrow[r, "\mathrm{loc}_n"] & H^1_f(G_{\mathbb{Q}_p}; U_n) \arrow[r, "\log_{\mathrm{BK}}", "\simeq"'] & U_n^{\mathrm{dR}}/F^0
\end{tikzcd}$$

and "Chabauty–Kim loci":

$$\mathcal{X}(R) \subseteq \mathcal{X}(\mathbb{Z}_p)^R_{CK,n} := \kappa_p^{-1}(\mathrm{Im}(\mathrm{loc}_n)) \subseteq \mathcal{X}(\mathbb{Z}_p)$$

Effectivity via Chabauty–Kim

Problem: Effective Faltings–Siegel

Question: Given R and $\mathcal{X}$, can we find the finite set $\mathcal{X}(R)$?

Let $\mathcal{I}^R_{CK,n} = \mathcal{I}^R_{CK,n}(\mathcal{X})$ denote the ideal of $\mathrm{LocAnal}(\mathcal{X}(\mathbb{Z}_p))$ generated by

$$\{\kappa_p^\#(f) \mid f \circ \mathrm{loc}_n = 0\}$$

Then $\mathcal{X}(\mathbb{Z}_p)^R_{CK,n} = Z(\mathcal{I}^R_{CK,n})$

Conjecture (Kim et al., 2014)

Let $k = \mathbb{Q}$. For sufficiently large n, the analytic subspace $Z(\mathcal{I}^R_{CK,n})$ is precisely $\mathcal{X}(R)$.

This conjecture reduces effective Faltings–Siegel to the following:

Refined Problem (Chabauty-Kim Theory)

Compute $\mathcal{I}^R_{CK,n}$ for arbitrary n and R

The Problem of Chabauty–Kim

Recall:

$$\begin{tikzcd}
\mathcal{X}(R) \arrow[r, "\mathrm{loc}_{\mathcal{X}}"] \arrow[d, "\kappa"] & \mathcal{X}(\mathbb{Z}_p) \arrow[d, "\kappa_p"] \arrow[dr, "\int_{BC}"] & \\
H^1_{f,R}(G_k; U_n) \arrow[r, "\mathrm{loc}_n"] & H^1_f(G_{\mathbb{Q}_p}; U_n) \arrow[r, "\log_{\mathrm{BK}}", "\simeq"'] & U_n^{\mathrm{dR}}/F^0
\end{tikzcd}$$

$$\mathcal{X}(\mathbb{Z}_p)^R_{CK,n} := \kappa_p^{-1}(\mathrm{Im}(\mathrm{loc}_n))$$

- The map $\int_{BC}$ is given by Coleman integration (for which explicit algorithms were given by Kedlaya, Balakrishnan, etc.)
- Key: understand $\mathrm{Im}(\mathrm{loc}_n)$
- More precisely, this requires a good understanding of the *Selmer variety* $H^1_{f,R}(G_k; U_n)$ and the *localization map* loc_n

Examples and Past Progress

- For $n = 1$, this is *classical Chabauty(–Skolem)*: for example, if $\mathcal{X}$ is a smooth projective curve with Jacobian J, then

$$H^1_{f,R}(G_k; U_1) = \mathrm{Sel}_{\mathbb{Q}_p}(J) \stackrel{?}{=} J(k) \otimes \mathbb{Q}_p$$

$$H^1_f(G_{\mathbb{Q}_p}; U_1) = J(\mathbb{Q}_p) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p$$

- The case $n = 2$ is covered mostly by *Quadratic Chabauty*, which understands the Selmer variety and localization map in terms of p-adic heights

For $n > 2$, much less is known:

- Until recently, the only computations for $n > 2$ were for $\mathcal{X}$ rational (e.g., an open subscheme of $\mathbb{P}^1$), beginning with work of Dan-Cohen–Wewers
- More recently, progress has been made for $n = 3$ and $\mathcal{X}$ a punctured elliptic curve; both by Balakrishnan–Bianchi–Dogra and by myself and Dan-Cohen and Lüdtke; the latter is the subject of this talk

The Problem of Understanding Selmer Varieties

Recall: the Selmer variety('s $\mathbb{Q}_p$-points) is $H^1_{f,R}(G_k; U_n)$. How can we understand it?

- Let $\mathrm{Rep}_{\mathbb{Q}_p}(G_k)$ denote the category of finite-dimensional $\mathbb{Q}_p$-vector spaces with continuous G_k-action
- U_n is a unipotent group over $\mathbb{Q}_p$ with continuous G_k-action;
- This means that $\mathcal{O}(U_n)$ is a Hopf algebra in the tensor category $\mathrm{Ind}\,\mathrm{Rep}_{\mathbb{Q}_p}(G_k)$
- $H^1(G_k; U_n)$ denotes the set of isomorphism classes of schemes over $\mathbb{Q}_p$ with continuous G_k-action and simply-transitive algebraic G_k-equivariant U_n-action
- $H^1_{f,R}(G_k; U_n)$ refers to those unramified over $\mathrm{Spec}\, R[1/p]$ and crystalline at places above p
- Problem: G_k and "crystalline" are complicated

Solution: Tannakian Selmer Varieties

- Idea: $H^1(G_k; U_n)$ can be described algebraically in terms of the tensor category $\mathrm{Ind}\,\mathrm{Rep}_{\mathbb{Q}_p}(G_k)$:
- the set of nonzero algebras A in $\mathrm{Ind}\,\mathrm{Rep}_{\mathbb{Q}_p}(G_k)$ with a coaction $\Delta\colon A \to A \otimes \mathcal{O}(U_n)$ for which $A \otimes A \xrightarrow{\mathrm{id}_A \otimes \Delta} A \otimes \mathcal{O}(U_n)$ is an isomorphism
- $H^1_{f,R}(G_k; U_n)$ can similarly be described algebraically in terms of the tensor category $\mathrm{Ind}\,\mathrm{Rep}^{f,R}_{\mathbb{Q}_p}(G_k)$
- Remark: all such torsors are in $\mathrm{Rep}^{f,R}_{\mathbb{Q}_p}(G_k, J)$, the thick Tannakian subcategory thereof generated by $V_p(J)$
- $\mathrm{Rep}^{f,R}_{\mathbb{Q}_p}(G_k, J)$ is not as hard to describe:
  - Its semisimple objects form the category of representations of the p-adic monodromy group $\mathbb{G} = \mathbb{G}(J)$ (close to the Mumford-Tate group $\mathrm{MT}(J)$)
  - Its Ext^1's are Bloch-Kato Selmer groups (whose dimensions are predicted by the Bloch-Kato conjectures)
  - Ext^i (conjecturally) vanishes for $i \geq 2$

Solution: Tannakian Selmer Varieties, cont.

- Choosing $\mathfrak{p}$ over p, we get a fiber functor $\mathrm{Rep}^{f,R}_{\mathbb{Q}_p}(G_k) \to \mathrm{Vect}(\mathbb{Q}_p)$ given by
$$V \mapsto \mathrm{D}_{\mathrm{dR}}(V)$$
- Tannakian formalism says: there exists a pro-algebraic group $\pi_1(R, J) = \pi_1^{\mathrm{dR}}(R, J)$ over $k_{\mathfrak{p}}$ for which
$$\mathrm{Rep}^{f,R}_{\mathbb{Q}_p}(G_k, J) \cong \mathrm{Rep}(\pi_1(R, J))$$
- Description of $\mathrm{Rep}^{f,R}_{\mathbb{Q}_p}(G_k, J)$ implies $\pi_1(R, J)$ is a (non-canonically) split extension of $\mathbb{G}(J)$ by a pro-unipotent group $U(R, J)$ generated by Bloch–Kato Selmer groups
- By the Tannakian formalism, $H^1_{f,R}(G_k; U_n^{\text{ét}})$ is the same as
$$H^1(\pi_1(R, J); U_n^{\mathrm{dR}})$$
- A generalization of a lemma of Dan-Cohen–Wewers implies:
$$H^1(\pi_1(R, J); U_n^{\mathrm{dR}}) = Z^1(U(R, J); U_n^{\mathrm{dR}})^{\mathbb{G}(J)}$$

History of the Idea

- The original approach of Dan-Cohen–Wewers uses the same idea: but here, instead of $\mathrm{Rep}^{f,R}_{\mathbb{Q}_p}(G_k, J)$, one uses a $\mathbb{Q}$-form: the $\mathbb{Q}$-linear Tannakian category of mixed Tate motives (c.f. also C.–Dan-Cohen)
- Problem: Tannakian categories of mixed motives are not known to exist[ii] beyond the mixed (Artin–)Tate case; this corresponds to $\mathcal{X}$ rational
- Both C. along with O. Patashnick and I. Dan-Cohen along with J. Cao were discussing using elliptic motives to cover more cases; later C. was working with I. Dan-Cohen
- C. realized: one could use $\mathbb{Q}_p$-rational categories of Galois representations unconditionally ($\implies$ `https://arxiv.org/abs/2102.08371`)
- This led to a relatively concrete theorem for a punctured elliptic curve for $n = 3$, described on Slide 16.

[ii] We don't have a t-structure!

History of the Idea, cont.

- The result depends on key technical work joint with I. Dan-Cohen: this allows one to describe loc_n and applies also to categories of motives
- With M. Lüdtke, I have carried out computations for specific elliptic curves
- Unbeknownst to us, Balakrishnan–Bianchi–Dogra had been working for a few years on the case of $n = 3$ for a punctured elliptic curve; their paper came out in April 2026
- While not phrased in Tannakian terms, a number of similar structures as in our work appear; my guess is that the approaches are closely related if not equivalent

Localization and Universal Cocycle Evaluation

The approach of C–Dan-Cohen in the mixed Tate case suggests that $\log_{BK} \circ \mathrm{loc}_n$ is given by evaluation at a special point $\mathrm{per}_p \in U(R,J)(\mathbb{Q}_p)$:

$$Z^1(U(R,J); U_n^{\mathrm{dR}})^{\mathbb{G}} \to U_n^{\mathrm{dR}} \twoheadrightarrow U_n^{\mathrm{dR}}/F^0 U_n^{\mathrm{dR}}$$

- This map sends $c \in Z^1(U(R,J); U_n^{\mathrm{dR}})^{\mathbb{G}}$ to $(c(\mathrm{per}_p)) \in U_n^{\mathrm{dR}}$.
- The map $\mathrm{per}_p \colon \operatorname{Spec} \mathbb{Q}_p \to U(R,J)$ was defined in the mixed Tate case by Chatzistamatiou–Ünver
- C–Dan-Cohen defines per_p in general an abstract setting that works for any "weight-filtered Tannakian category with Hodge-filtered and Frobenius-equivariant fiber functors"
- e.g., strongly geometric crystalline Galois representations, mixed Motives, systems of realizations
- We explain per_p more concretely on the next slide: (IF TIME)

The p-adic period map

- The coordinate ring of $U(R, J)$ is a Hopf algebra we call

$$A(R, J)$$

- For $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, it is graded; for $X = E'$, it has a GL_2-action; more generally, an action of $\mathbb{G}$
- The map per_p is equivalently a ring homomorphism $\mathrm{per}_p \colon A(R, J) \to \mathbb{Q}_p$
- Elements of $A(R, J)$ can be seen as *(de Rham) motivic periods* (sense of Grothendieck, Kontsevich–Zagier, Brown), and per_p sends them to their p-adic (Coleman) period
- Our result relates per_p to $\log_{BK} \circ \mathrm{loc}_n$

Theorem of C–Dan-Cohen

Theorem (C–Dan-Cohen, 2026)

There is a point

$$\mathrm{per}_p\colon \operatorname{Spec}\mathbb{Q}_p \to U(R,J),$$

satisfying the following property:
For all

$$c \in Z^1(U(R,J), U_n^{\mathrm{dR}})^{\mathbb{G}}(\mathbb{Q}_p),$$

the composition

$$c/F^0 \circ \mathrm{per}_p\colon \operatorname{Spec}\mathbb{Q}_p \xrightarrow{\mathrm{per}_p} U(R,J) \xrightarrow{c} U_n^{\mathrm{dR}} \twoheadrightarrow U_n^{\mathrm{dR}}/F^0 U_n^{\mathrm{dR}}$$

is $\log_{BK}(\mathrm{loc}_n(c)) \in U_n^{\mathrm{dR}}/F^0 U_n^{\mathrm{dR}}(\mathbb{Q}_p)$.

The explicit result of (C, 2021) on Slide 16 depends on this theorem.

General Method

- The result of Slide 14 reduces computation of $\mathcal{I}^R_{CK,n}$ to computing a basis of $A(R,J)$ on which we know per_p
- We can actually use the theorem to do this: given $z \in X$ and $\omega \in \mathcal{O}(U_n^{\mathrm{dR}}/F^0 U_n^{\mathrm{dR}})$, we can pull back ω along $\kappa(z)$ to get an element of $A(R,J)$ denoted $I^{\mathfrak{m}}(b:\omega:z)$ known as a *motivic iterated integral*
- We then have
$$\mathrm{per}_p(I^{\mathfrak{m}}(b:\omega:z)) = \int_b^z \omega,$$
an iterated Coleman integral.
- More generally, if we replace X by any variety whose unipotent étale fundamental group lies in $\mathrm{Rep}^{\mathrm{f,R}}_{\mathbb{Q}_p}(G_{\mathbb{Q}}, J)$, we get more motivic iterated integrals
- The subspace $A(R,J)^{\mathrm{prim}} := \{\alpha \in A(R,J) \mid \Delta\alpha = \alpha \otimes 1 + 1 \otimes \alpha\}$ is closely related to extensions of motives and receives maps from K-theory
- The restriction of per_p to $A(R,J)^{\mathrm{prim}}$ is the syntomic regulator

Chabauty-Kim for a Punctured Elliptic Curve

- Set $\eta_0 := \frac{dx}{y}$ and $\eta_1 = \frac{xdx}{y}$
- Set $J_1 := \int \eta_0$, $J_2 := \int \eta_0\eta_1$, $J_3 := \int \eta_0\eta_1\eta_0$, and $J_4 := \int \eta_0\eta_1\eta_1 + 2\int \eta_1$

Theorem (C, 2021)+(C–Dan-Cohen, 2026)

If $\ell \neq p$ are primes and E is an elliptic curve over $\mathbb{Q}$ of p-Selmer rank 1 with good ordinary reduction at p and $\mathcal{L}_p(E,2) \neq 0$, then there is a function of the form

$$c_1 J_4 + c_2 J_3 + c_3 J_1 J_2 + c_4 J_1^3 + c_5 J_1$$

in $\mathcal{I}_{CK,3}^{\mathbb{Z}[1/\ell]}(\mathcal{X})^a$, with $c_i \in \mathbb{Q}_p$ not all zero.

[a] $\mathcal{X} = \mathcal{E} \setminus \{O\}$, $\mathcal{E}$ a good integral model of E

We now describe concrete computations with M. Lüdtke

Recent concrete computations (with M. Lüdtke)

With Martin Lüdtke, we have computed this function and $Z(\mathcal{I}_{CK,3}^{\mathbb{Z}[1/2]})$ for $E = 128a2$ and $p = 3, 5, 7$ and $E = 102a1$ for $p = 5, 7$[iii]

Example

For E the elliptic curve $y^2 = x^3 + x^2 - 9x + 7$ (Cremona label '128a2'), $R = \mathbb{Z}[1/2]$, and $p = 3, 5$:

	$p = 3$	$p = 5$
c_1	1	1
c_2	$123659722/3^2 + O(3^{15})$	$202228537903/5^2 + O(5^{15})$
c_3	$113658899/3^2 + O(3^{15})$	$106941634807/5^2 + O(5^{15})$
c_4	$667280621/3^4 + O(3^{15})$	$2783496064817/5^3 + O(5^{15})$
c_5	$65854279/3^2 + O(3^{15})$	$64558043327/5 + O(5^{15})$

[iii] using code of Jennifer Balakrishnan for computing triple Coleman integrals

Recent concrete computations (with M. Lüdtke), cont.

The Chabauty–Kim locus $\mathcal{X}(\mathbb{Z}_p)^R_{CK,3}$ consists of the 13 known points

$$(-3,-4),(-3,4),(-1,-4),(-1,4),(1,0),(2,-1),(2,1),(3,-4),(3,4),$$
$$(29/4,-155/8),(29/4,155/8),(19,-84),(19,84),$$

along with the additional p-adic points with x-coordinates as follows:

$p=3$	$p=5$	$p=7$
$5830650+O(3^{15})$	$166703174+O(5^{15})$	$-1+2\sqrt{2}$
	$1482191568+O(5^{15})$	$-1-2\sqrt{2}$
	$9650625907+O(5^{15})$	$574758317373+O(7^{15})$
	$25074712578+O(5^{15})$	$1162578601155+O(7^{15})$
		$2351419738405+O(7^{15})$
		$2597613053696+O(7^{15})$
		$3582802419202+O(7^{15})$
		$4286473007874+O(7^{15})$

Useful References/Credits

- Explicit Motivic Mixed Elliptic Chabauty-Kim, David Corwin, arXiv
- Tannakian Selmer Varieties, David Corwin, website (`math.bgu.ac.il/~corwind`)
- *p*-Adic Periods and Selmer Scheme Images, David Corwin and Ishai Dan-Cohen, arXiv
- Computations in mixed elliptic motivic Chabauty–Kim: Initial Report, David Corwin and Martin Lüdtke, `https://martinluedtke.github.io/files/corwin_luedtke_initial_report_2026.pdf`

Published:

- Mixed Tate Motives and the Unit Equation, Ishai Dan-Cohen and Stefan Wewers
- Mixed Tate Motives and the Unit Equation II, Ishai Dan-Cohen
- The polylog quotient and the Goncharov quotient in computational Chabauty-Kim theory I, David Corwin and Ishai Dan-Cohen
- The polylog quotient and the Goncharov quotient in computational Chabauty-Kim theory II, David Corwin and Ishai Dan-Cohen

Thank You!