Towards coarse moduli of augmentations

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Work in slow progress.

Outline

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- Let $Z \subset \operatorname{Spec} \mathcal{O}_K$ open.[1]
- Let $\mathcal{D}_M = \mathrm{DM}(Z, \mathbb{Q})$.
- Preview:
  - A variety X over Z gives rise to a highly structured algebra $C^*(X)$ in $\mathcal{D}_M$.
  - A Z-point gives rise to an augmentation $C^*(X) \to \mathbb{1}$.
  - This assignment $X(Z) \to \operatorname{Aug}\left(C^*(X)\right)$ factors the unipotent Kummer map.
  - This suggests the possibility of *performing* Chabauty-Kim theory motivically without waiting for a t-structure.
  - In a different (largely independent) direction, this may allow us to extract arithmetic information from the full rational homotopy type going beyond π_1, e.g. for higher dimensional varieties with small π_1 and (conjecturally) degenerate set of integral points.
  - In both directions, $\operatorname{Aug}\left(C^*(X)\right)$ would benefit from a structure of finite type $\mathbb{Q}$-variety. I'll show preliminary results and examples in this direction.

[1] At least in most of our examples.

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Properties of $\mathcal{D}_M$:

- It's an ∞-category, so given $E, F \in \mathcal{D}_M$, there's an associated object $\operatorname{Hom}_{\mathcal{D}_M}(E, F)$ of the ∞-category $\mathcal{H}$ of homotopy types; may be modeled by a topological space.
- There's an associated 1-category $h\mathcal{D}_M$ in which

$$\operatorname{Hom}_{h\mathcal{D}_M}(E, F) = \pi_0 \operatorname{Hom}_{\mathcal{D}}(E, F).$$

- More properties:
  - It's *stable*. So there's a zero-object $0 \in \mathcal{D}_M$; a square diagram is Cartesian iff it's cocartesian; given $E \in \mathcal{D}_M$, the *suspension* and *loop space* are defined by the following (homotopy) pullback and pushout:

$$\begin{tikzcd} \Omega E \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & E \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma E; \end{tikzcd}$$

    the pairs of Cartesian squares

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & G \arrow[r] & \Sigma E \end{tikzcd}$$

    give $h\mathcal{D}_M$ the structure of a triangulated category.

- More properties
  - It's $\mathbb{Q}$-linear: the groups

$$\pi_i \operatorname{Hom}_{\mathcal{D}_M}(E,F) = \pi_0 \Omega^i \operatorname{Hom}_{\mathcal{D}_M}(E,F) = \pi_0 \operatorname{Hom}_{\mathcal{D}_M}(E, \Omega^i F)$$

    have the structure of $\mathbb{Q}$-*vector spaces*.
  - It's presentably symmetric monoidal.
  - Let $\mathcal{C}_M = \operatorname{CAlg} \mathcal{D}_M$. There are functors[2]

$$\mathrm{Sm}_Z \xrightarrow{C_*} \mathcal{D}_M$$

$$\begin{tikzcd} \mathrm{Sm}_Z^{\mathrm{op}} \arrow[r, "C^*"] \arrow[dr, "C^* = C_*^\vee"'] & \mathcal{C}_M \arrow[d] \\ & \mathcal{D}_M. \end{tikzcd}$$

  - For $X \in \mathrm{Sm}_Z$, there are isomorphisms

$$\pi_0 \operatorname{Hom}_{\mathcal{D}_M}\left(C_*(X), \mathbb{Q}(i)(j)\right) \simeq K_{2i-j}^{(i)}(X).$$

[2] The functor C_* can equally be upgraded to coalgebras.

- More properties
  - There's a W-realization functor W^* associated to any suitable Weil cohomology theory, e.g. given a complex embedding $K \subset \mathbb{C}$, there's an equivalence
$$\mathbb{C} \otimes_{\mathbb{Q}} \mathrm{Betti}^* C^*(X) \simeq \Omega^\bullet_{\mathrm{smooth}}\big(X(\mathbb{C})\big)$$
in $\mathrm{CAlg}\,\mathcal{D}(\mathbb{C}) \simeq \mathrm{cdga}_{\mathbb{C}}$; or if X is affine, then
$$\mathrm{dR}^* C^*(X) \simeq \Omega^\bullet_{\mathrm{alg}}(X_K)$$
in cdga_K.

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We have $C^*(Z) = \mathbb{1}$ (the tensor unit $\mathbb{Q}(0)$ with its essentially unique algebra structure).
By functoriality, a Z-point $x \in X(Z)$ gives rise to an augmentation

$$C^*(X) \xrightarrow{x} \mathbb{1}.$$

Definition

- *Given* $C \in \mathcal{C}_M$, *we consider an associated presheaf of homotopy types*

$$\mathcal{A}\mathrm{ug}(C) : \mathrm{Aff}_{\mathbb{Q}}^{\mathrm{op}} \to \mathcal{H}$$

  with values

$$\mathcal{A}\mathrm{ug}(C)(R) = \mathsf{Hom}_{\mathcal{C}_M}(C, R \otimes \mathbb{1})$$

  on any $\mathbb{Q}$*-algebra* R*.*

- *For* $X \in \mathsf{Sm}_Z$, *we let*

$$\mathcal{A}\mathrm{ug}(X) := \mathcal{A}\mathrm{ug}\left(C^*(X)\right).$$

Problem

Understand when is the presheaf of connected components $\pi_0\,\mathcal{A}\mathrm{ug}(X)$ *representable by a finite type* $\mathbb{Q}$*-variety.*

Answer: Sometimes Yes. For instance,

Theorem (Follows from work of Ancona et. al.)

Let $A \to Z$ be a semiabelian scheme. Then [a]

$$\mathcal{A}\mathrm{ug}(A) \simeq \mathbb{V}^\vee A(Z)_{\mathbb{Q}}.$$

[a] $\mathbb{V}^\vee(-) = \operatorname{Spec}\operatorname{Sym}(-)^\vee$.

And also,

Theorem A (Direct consequence of joint work with L. A. Betts)

Let $L^* \to A \to Z$ be a $\mathbb{G}_m$-bundle over a semiabelian scheme. Then the augmentation space is a trivial torsor:

$$\begin{array}{ccc} \mathbb{V}^\vee \mathbb{G}_m(Z)_{\mathbb{Q}} & \circlearrowright & \mathcal{A}\mathrm{ug}(L^*) \\ & & \downarrow \\ & & \mathbb{V}^\vee A(Z)_{\mathbb{Q}}. \end{array}$$

Remark

Theorem A is based on a determination[a] *of the structure of* $C^*(L^*)$. *The latter instantiates the expectation that highly structured algebras in motives with rational coefficients should form a reasonably computable flavor of unstable motivic homotopy.*

[a] Of some interest in its own right.

Proof later, time permitting.

Answer: But usually probably very infinitary.
Solution: Use a filtration / tower studied by I. Iwanari.

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Given $C \in \mathcal{C}_M$, define the tower on the left...

$$
\begin{tikzcd}
C & \\
\vdots \arrow[u] & \\
\mathfrak{I}_2 C \arrow[u] & \mathrm{Sym}\, V_2 \arrow[l] \\
\mathfrak{I}_1 C \arrow[u] & \mathrm{Sym}\, V_1 \arrow[l] \\
\mathbb{1} \arrow[u] & \mathrm{Sym}\, V_0 \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
V_n \arrow[r] & \mathfrak{I}_n C \arrow[r] & C \arrow[r] & V_n[1]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathfrak{I}_{n+1} C & \mathbb{1} \arrow[l] \\
\mathfrak{I}_n C \arrow[u] & \mathrm{Sym}\, V_n \arrow[l] \arrow[u]
\end{tikzcd}
$$

by the exact triangles in $\mathcal{D}_M$ and the pushout squares in $\mathcal{C}_M$ on the right.

Definition

- $\mathcal{A}\mathrm{ug}^n(C) = \mathcal{A}\mathrm{ug}(\mathfrak{I}_n C)$
- $\mathcal{A}\mathrm{ug}^n(X) = \mathcal{A}\mathrm{ug}\left(\mathfrak{I}_n C^*(X)\right)$.

Variant

- $\mathcal{A}\mathrm{ug}^n_W(X) = \mathcal{A}\mathrm{ug}\left(W^* \mathfrak{I}_n C^*(X)\right)$.

Outline of talk:

- Compatibility with Chabauty-Kim
- Coarse criterion for representability
- Example: X a punctured elliptic scheme, $n = 3$.
- Refined criterion

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Assume $X \to Z$ is the complement of an étale divisor in a smooth proper scheme.
Fix a Z-integral base-point x.
Let $W = p$-adic étale cohomology with Galois action, or mixed Hodge theory.

Theorem (Long history, including joint work with T. Schlank, joint work with L. A. Betts, plus work of Tur-Dorvault in the case of tangential base points)

$$\mathrm{Spec}\, H^0 W^* \mathrm{Bar}\big(C^*(X) \xrightarrow{x} \mathbb{1}\big) = \pi_1^{\mathrm{un}}(X, x)_W$$

is the W-realization of the unipotent fundamental group compatibly with additional structures.

Example: $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

- $C^*(X) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)[-1]^{\oplus 2}$.
- The multiplication is stupid.
- But the higher coherences remember the entire unipotent fundamental group!

Remark

A similar statement holds for path torsors.

Theorem B (Compatibility with CK)

For simplicity, let $Z \subset \operatorname{Spec}\mathbb{Z}$ *(*$K = \mathbb{Q}$*). Fix* $p \in Z$*. Fix* n *and let* $\pi_1^{[n]}(X, x)$ *denote the nth quotient along the dcs. Then there's a commutative diagram*

$$\begin{tikzcd}
X(Z) \arrow[r] \arrow[d] & X(\mathbb{Z}_p) \arrow[d] \\
\pi_0\,\mathcal{A}\mathrm{ug}^n(X)_{\mathbb{Q}_p} \arrow[r] \arrow[d] & \pi_0\,\mathcal{A}\mathrm{ug}^n_{F\phi}(X_{\mathbb{Q}_p}) \arrow[d] \\
H^1_f\big(\pi_1^{\acute{e}t}(Z \setminus \{p\}), \pi_1^{[n]}(X,x)^{\acute{e}t}_{\mathbb{Q}_p}\big) \arrow[r] & F^0\backslash \pi_1^{[n]}(X,x)^{\mathrm{dR}}_{\mathbb{Q}_p}
\end{tikzcd}$$

factoring Kim's cutter. The lower square consists of maps of presheaves.

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Consider the Iwanari tower and the dual tower of presheaves of augmentations ($\mathcal{A}^n = \mathcal{A}\mathrm{ug}^n(C)$):

$$
\begin{tikzcd}
C & & & \mathcal{A} \arrow[d] & \\
\vdots \arrow[u] & & & \vdots \arrow[d] & \\
\mathfrak{I}_2 C \arrow[u] & \mathrm{Sym}\, V_2 \arrow[l] & & \mathcal{A}^2 \arrow[r] \arrow[d] & \mathcal{V}^2 \\
\mathfrak{I}_1 C \arrow[u] & \mathrm{Sym}\, V_1 \arrow[l] & & \mathcal{A}^1 \arrow[r] \arrow[d] & \mathcal{V}^1 \\
\mathbb{1} \arrow[u] & \mathrm{Sym}\, V_0 \arrow[l] & & * \arrow[r] & \mathcal{V}^0
\end{tikzcd}
\tag{*}
$$

Theorem C

Assume

- V_i *is a sum of compact objects for all* $0 \leq i \leq n-1$,
- $\pi_2 \mathcal{V}^i(\mathbb{Q}) = 0$ *for all* $0 \leq i \leq n-2$, *and*
- $\pi_1 \mathcal{V}^i(\mathbb{Q})$, $\pi_0 \mathcal{V}^i(\mathbb{Q})$ *are finite dimensional for all* $0 \leq i \leq n-1$.

Then

1. $\pi_0 \mathcal{A}^n$ *is representable by a finite-type affine* $\mathbb{Q}$*-scheme,*
2. *for every* $\mathbb{Q}$*-algebra* R, *every connected component of* $\mathcal{A}^{n-1}(R)$ *is simply connected, and*
3. *we have*

$$\dim \pi_0 \mathcal{A}^n \leq \sum_{i=0}^{n-1} \dim \pi_1 \mathcal{V}^i(\mathbb{Q}),$$

with equality if $\pi_0 \mathcal{V}^i(\mathbb{Q}) = 0$ *for* $i = 1, \ldots, n-1$.

Example (sketch): $X = E \setminus \{0\}$ a punctured elliptic curve over $Z \subset \operatorname{Spec}\mathbb{Z}$. Iwanari computes $\mathfrak{I}_3 C = C^*(X)$ as follows. The portion $\mathfrak{I}_{\leq 2}$ of the tower (solid arrows):

$$
\begin{tikzcd}
 & C = \mathbb{Q}(0) \oplus h^1(E)[-1] & \\
 & \vdots \arrow[u] & \\
 & \mathfrak{I}_2 C \arrow[u] & \mathbb{1} \arrow[l, dotted] \\
\mathbb{Q}(0) \oplus h^1(E)[-1] \oplus \mathbb{Q}(-1)[-2] \arrow[r, equal] & \operatorname{Sym} h^1(E)[-1] \arrow[u] & \operatorname{Sym} \mathbb{Q}(-1)[-2] \arrow[l] \arrow[u, dotted] \\
 & \mathbb{1} \arrow[u] &
\end{tikzcd}
$$

with $\mathfrak{I}_2 C$ given by a pushout (dotted arrows).

We have, $\wedge^3 h^1(E) = 0$. It follows from Tannakian considerations obtained for instance by Iwanari that there's a symmetric monoidal colimit preserving functor

$$\mathcal{D}\,\mathrm{Rep}_{\mathbb{Q}}\,\mathrm{GL}_2 \to \mathcal{D}_M$$

Sending

$$V_{\text{standard}} \mapsto h^1(E).$$

Hence, the pushout is given by a GL_2-equivariant Hirsch extension. We find that

$$\mathfrak{I}_2 C \simeq \mathbb{Q}(0) \oplus h^1[-1] \oplus V_2$$

where

$$V_2 = H^1(-1)[-2] \oplus \mathbb{Q}(-2)[-3].$$

Gathering what we know about the corresponding K-groups, we find that the lower portion of the tower and dual tower are given by

$$
\begin{tikzcd}
\mathfrak{J}_3 C & \\
\mathfrak{J}_2 C \arrow[u] & \mathrm{Sym}\left(H^1(-1)[-2] \oplus \mathbb{Q}(-2)[-3]\right) \arrow[l] \\
\mathrm{Sym}\left(h^1(E)[-1]\right) \arrow[u] & \mathrm{Sym}\left(\mathbb{Q}(-1)[-2]\right) \arrow[l] \\
\mathbb{1} \arrow[u] & \mathrm{Sym}\left(h^1(E)[-2]\right) \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}^3 \arrow[d] & \\
\mathcal{A}^2 \arrow[d] \arrow[r] & K^{(2)}_{\bullet+1}(X) \\
E(Z)_{\mathbb{Q}} \arrow[d] \arrow[r] & K(\mathcal{O}^*_{Z,\mathbb{Q}}, 1) \\
* \arrow[r] & K\left(E(Z)_{\mathbb{Q}}, 1\right).
\end{tikzcd}
$$

The most relevant homotopy groups are as follows:

$$
\begin{tikzcd}
* \arrow[r] & K_2^{(2)}(X) & \circlearrowright & \pi_0 \mathcal{A}^3 \arrow[d] & \\
* \arrow[r] & \mathcal{O}^*_{Z,\mathbb{Q}} & \circlearrowright & \pi_0 \mathcal{A}_2 \arrow[r] \arrow[d] & * \\
 & & & E(Z)_{\mathbb{Q}} \arrow[r] \arrow[d] & * \\
 & & & * &
\end{tikzcd}
$$

We obtain a motivic, potentially effective, proof of the following special case of Siegel's theorem. Let $s = \sharp(\operatorname{Spec} \mathbb{Z} \setminus Z)$. Let $r = \operatorname{rk} E(Z)$. If

$$s + r + k_2^{(2)}(X) < 4,$$

then $X(Z)$ is finite.

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Assume $\pi_0\mathcal{A}^n$, $\pi_0\mathcal{V}^n$ are representable. Let

$$L_n \subset \pi_0\mathcal{A}^n \to \pi_0\mathcal{V}^n$$

be the fiber above 0. Then $\mathcal{A}^n|_{L_n}$ admits an L_n-valued basepoint x over the identity map of L_n, and there's an induced map of presheaves of homotopy groups over L_n (solid arrows)

$$\begin{tikzcd} \pi_1(\mathcal{A}^n|_{L_n}, x) \arrow[r] \arrow[dr] & \pi_1(L_n \times \mathcal{V}^n, 0) \arrow[r, dotted] \arrow[d] & Q_n \arrow[dl, dotted] \\ & L_n & \end{tikzcd}$$

Let Q_n be the cokernel.

Proposition D

The map $\pi_0\mathcal{A}^{n+1} \to L_n$ is a trivial Q_n-torsor.

So if Q_n is representable, so is $\pi_0\mathcal{A}^{n+1}$.

Example (sketch): $Z \subset \operatorname{Spec} \mathcal{O}_K$.
$X = \mathbb{A}^2 \setminus \{x = 0, y = 0, \text{ or } x + y = 1\}$. Let $\mathbb{E}_i = \operatorname{Ext}^1_Z\left(\mathbb{Q}(0), \mathbb{Q}(i)\right)$.
The lower portion of the Iwanari tower and the dual tower take the following form:

$$\begin{tikzcd}
\mathbb{Q}(0) \oplus \mathbb{Q}(-1)[-1]^{\oplus 3} \oplus \mathbb{Q}(-2)[-2]^{\oplus 3} & \vdots \arrow[d] & \\
\vdots \arrow[u] & \mathcal{A}^2 \arrow[d] & \\
\mathfrak{I}_2 C \arrow[u] & K(\mathbb{E}_1, 0)^3 \times K(\mathbb{E}_2, 1)^3 \arrow[r] & K(\mathbb{E}_2, 1)^3 \times \mathcal{V}' \\
\mathrm{Sym}\left(\mathbb{Q}(-1)[-1]^{\oplus 3} \oplus \mathbb{Q}(-2)[-2]^{\oplus 3}\right) \arrow[u] & \mathrm{Sym}\left(\mathbb{Q}(-2)[-2]^{\oplus 3} \oplus V'\right) \arrow[l] &
\end{tikzcd} \tag{*}$$

with V' a sum of shifts of Tate motives $\mathbb{Q}(-i)[-i]$ with $i \geq 2$, so that $\mathcal{V}'$ is connected and simply connected.

We find that Q_1 is the cokernel of a map of trivial vector groups of rank 3:

$$
\begin{tikzcd}
\mathbb{E}_1^3 \times \mathbb{E}_2^3 \arrow[r, "\psi"] & \mathbb{E}_1^3 \times \mathbb{E}_2^3 \arrow[r] & Q_1 \arrow[r, phantom, "\circlearrowright"] & \pi_0 \mathcal{A}^2 \arrow[d, twoheadrightarrow, "\phi"] \\
& & & \mathbb{E}_1^3.
\end{tikzcd}
$$

Proposition

The map ψ is an isomorphism, so $Q_1 = 0$ and ϕ is an isomorphism.

We've found that $\pi_0\, \mathcal{A}\mathrm{ug}^2(X)$ is representable, albeit not very interesting.

Steps towards an iterative *method*. Assume the unit $\mathbb{1} \to C$ splits in $\mathcal{D}_M$. The exact couple of homotopy presheaves (e.g. over L_3) may be arranged into a grid of knights moves:

$$
\begin{tikzcd}
\pi_3\mathcal{A}^1 \arrow[r, squiggly, "\alpha_3"] & \pi_3\mathcal{V}^1 \arrow[r, squiggly] & \pi_2\mathcal{A}^2 \arrow[r, "\alpha_2"] \arrow[d, squiggly] & \pi_2\mathcal{V}^2 \arrow[r] & \pi_1\mathcal{A}^3 \arrow[r, "\alpha_1"] \arrow[d] & \pi_1\mathcal{V}^3 \arrow[r] & \pi_0\mathcal{A}^4 \arrow[r] \arrow[d] & \pi_0\mathcal{V}^4 \\
& & \pi_2\mathcal{A}^1 \arrow[r, squiggly] & \pi_2\mathcal{V}^1 \arrow[r, squiggly] & \pi_1\mathcal{A}^2 \arrow[r] \arrow[d, squiggly] & \pi_1\mathcal{V}^2 \arrow[r] & \pi_0\mathcal{A}^3 \arrow[r] \arrow[d] & \pi_0\mathcal{V}^3 \\
& & & & \pi_1\mathcal{A}^1 \arrow[r, squiggly] & \pi_1\mathcal{V}^1 \arrow[r, squiggly] & \pi_0\mathcal{A}^2 \arrow[r] \arrow[d, squiggly] & \pi_0\mathcal{V}^2 \\
& & & & & & \pi_0\mathcal{A}^1 \arrow[r, squiggly] \arrow[d] & \pi_0\mathcal{V}^1 \\
& & & & & & * &
\end{tikzcd}
$$

"Unipotent" will mean *representable by a unipotent group scheme over L_3.* (Most of them will be vector groups.)

- We assume the homotopy groups $\pi_{\geq 1}$ below the top row have already been shown to be unipotent.
- We assume $\pi_0 \mathcal{A}_3$ is representable.

We then have a chain of weak implications:

$$
\begin{gathered}
\pi_3\mathcal{A}^1 \text{ unipotent (so } \alpha_3 \text{ is a map of unipotent groups)} \\
\rightsquigarrow \\
\operatorname{cok}\alpha_3 \text{ is unipotent} \\
\Downarrow \\
\pi_2\mathcal{A}^2 \text{ is unipotent (} \alpha_2 \text{ is a map of unipotent groups)} \\
\rightsquigarrow \\
\operatorname{cok}\alpha_2 \text{ is unipotent} \\
\Downarrow \\
\pi_1\mathcal{A}^3 \text{ is unipotent(} \alpha_1 \text{ is a map of unipotent groups)} \\
\rightsquigarrow \\
\operatorname{cok}\alpha_1 = Q_3 \text{ is unipotent} \\
\Downarrow \\
\pi_0\mathcal{A}^4 \text{ is representable.}
\end{gathered}
$$

Appendices

viii. The motivic cochain algebra of an abelian variety

ix. The motivic cochain algebra of a $\mathbb{G}_m$-torsor: statement

x. The motivic cochain algebra of a $\mathbb{G}_m$-torsor: proof

The de Rham dga of an abelian variety $A_{\mathbb{C}}$ over $\mathbb{C}$ is equivalent to the cdga with zero differential

$$C^*_{\mathrm{dR}}(A_{\mathbb{C}}) \simeq \mathrm{Sym}\, H^1_{\mathrm{dR}}(A_{\mathbb{C}}).$$

A similar statement holds motivically:

- Ancona et. al. construct $M_1(A) \in \mathcal{D}_M$ by considering the sheaf of $\mathbb{Q}$-vector spaces $A \otimes \mathbb{Q}$.
- Let $M^1 = M_1^\vee$.
- Iwanari shows that
$$C(A) = \operatorname{Sym} M^1(A).$$

Thus,

$$\mathcal{A}\mathrm{ug}(A) = \operatorname{Hom}_{\mathcal{D}_M}\left(M^1(A), \mathbb{Q}(0)\right) = \operatorname{Hom}_{\mathcal{D}_M}(\mathbb{Q}(0), M_1(A)) = A(K) \otimes \mathbb{Q}.$$

Appendices

If $Y_{\mathbb{C}}$ is smooth over $\mathbb{C}$ and $L_{\mathbb{C}} \to Y_{\mathbb{C}}$ is a line bundle with first Chern class $c_{\mathrm{dR}} \in H^2_{\mathrm{dR}}(Y_{\mathbb{C}})$ and associated $\mathbb{G}_m$-torsor L^*, then

$$\Omega^\bullet_{C^\infty}(L^*) \simeq \Omega^\bullet_{C^\infty}(Y)[t]/(dt - c_{\mathrm{dR}}).$$

More precisely, if

$$\mathbb{C}[-2] \xrightarrow{c_{\mathrm{dR}}} C^*_{\mathrm{dR}}(Y_{\mathbb{C}}) \to E$$

is the cofiber of c_{dR} then the induced square

$$\begin{tikzcd} \mathrm{Sym}\, C^*_{\mathrm{dR}}(Y_{\mathbb{C}}) \arrow[r] \arrow[d] & \mathrm{Sym}\, E \arrow[d] \\ C^*_{\mathrm{dR}}(Y_{\mathbb{C}}) \arrow[r] & C^*_{\mathrm{dR}}(L^*_{\mathbb{C}}) \end{tikzcd}$$

is a homotopy pushout square.

Now let Y be smooth over Z and L a line bundle over Y. Let

$$c^L : M^1(\mathbb{G}_m)[-1] = \mathbb{Q}(-1)[-2] \to C^*(Y)$$

be the motivic first Chern class, and let

$$\mathbb{Q}(-1)[-2] \to C^*(Y) \to E \to \mathbb{Q}(-1)[-1] = M^1(\mathbb{G}_m)$$

be the cofiber in $\mathcal{D}$.

Theorem E (Joint with L. A. Betts)

The square

$$\begin{tikzcd} \mathrm{Sym}\, C^*(Y) \arrow[r] \arrow[d] & \mathrm{Sym}\, E \arrow[d] \\ C^*(Y) \arrow[r] & C^*(L^*) \end{tikzcd}$$

is (homotopy) coCartesian.

- Hence, the augmentation space sits in a homotopy pullback square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}\left(C^*(Y), \mathbb{Q}(0)\right) & \mathrm{Hom}_{\mathcal{D}}\left(E, \mathbb{Q}(0)\right) \arrow[l] & \mathcal{A}\mathrm{ug}(\mathbb{G}_m) = \mathcal{O}_Z^* \otimes \mathbb{Q} \arrow[l] \\
\mathcal{A}\mathrm{ug}(Y) \arrow[u] & \mathcal{A}\mathrm{ug}(L^*) \arrow[l] \arrow[u] &
\end{tikzcd}$$

- It follows that $\mathcal{A}\mathrm{ug}(L^*)$ has the structure of an $\mathcal{O}_Z^* \otimes \mathbb{Q}$-torsor over $\mathcal{A}\mathrm{ug}(Y)$.

Appendices

Let P be the pushout

$$\begin{tikzcd} \operatorname{Sym} C^*(Y) \arrow[r] \arrow[d] & \operatorname{Sym} E \arrow[d] \\ C^*(Y) \arrow[r] & P. \end{tikzcd}$$

Then by base-changing everything to $C^*(Y)$ we find that P is also a pushout

$$\begin{tikzcd} \operatorname{Sym}_{C^*(Y)} C^*(Y) \arrow[r] \arrow[d] & \operatorname{Sym}_{C^*(Y)} C^*(L^*) \arrow[d] \\ C^*(Y) \arrow[r] & P. \end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Sym}_{C(Y)} C(Y) \arrow[r] \arrow[d] & \mathrm{Sym}_{C(Y)} C(L^*) \arrow[d] \\
C(Y) \arrow[r] & P.
\end{tikzcd}$$

The copy of $C(L^*)$ in the upper right comes from the Gysin sequence

$$C(Y)(-1)[-2] \xrightarrow{\text{mult. by } c^L} C(Y) \to C(L^*) \to C(Y)(-1)[-1]. \qquad (*)$$

This shows that P may be identified with the "relatively free commutative algebra" generated by the pointed object

$$C(Y) \to C(L^*)$$

of $\mathrm{Mod}_{C(Y)}$.

Lurie's theory of free algebras shows that

$$P = \operatorname{colim} C(L^*)^{c^L \otimes}$$

is given by the colimit of a diagram

$$C(L^*)^{c^L \otimes} : \mathrm{Fin}_{\mathrm{inj}} \to \mathrm{Mod}_{C(Y)}$$

which mixes the symmetric group actions on the tensor powers of $C(L^*)$ with multiplication by the 1st Chern class c^L. Using the fact that $\mathrm{Mod}_{C(Y)}$ is tensored over rational spaces, we may decompose the colimit as

$$P = \operatorname{colim}_n \mathrm{Sym}^n_{C(Y)} C(L^*).$$

Taking symmetric powers over $C(Y)$ in the exact triangle

$$C(Y)(-1)[-2] \xrightarrow{\text{mult. by } c^L} C(Y) \to C(L^*) \to C(Y)(-1)[-1], \qquad (*)$$

we get exact triangles

$$\mathrm{Sym}^{n-1}_{C(Y)} C(L^*) \to \mathrm{Sym}^n_{C(Y)} C(L^*) \to \mathrm{Sym}^n_{C(Y)} C(Y)(-1)[-1].$$

We have

$$\mathrm{Sym}^n_{C(Y)} C(Y)(-1)[-1] = 0 \quad \text{for } n \geq 2.$$

So finally,

$$P = \mathrm{colim}(C(Y) \to C(L^*) \xrightarrow{=} C(L^*) \xrightarrow{=} C(L^*) \xrightarrow{=} \cdots) = C(L^*).$$