Computations with hyperelliptic Mumford Curves: Uniformization, p-adic integrals, p-adic heights

Enis Kaya (Bilkent University)
joint work ~~in progress~~ with
Marc Masdeu, J. Steffen Müller and Marius van der Put

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§1.1. Introduction

In the 70's, Mumford discovered p-adic analogues of complex uniformizations of curves and abelian varieties, where p is a fixed prime number.

The curves that admit p-adic uniformization are called **Mumford curves**; their geometry and arithmetic are quite rich.

Goal

For a given hyperelliptic Mumford curve C, give algorithms to compute

- a p-adic uniformization of C,
- p-adic Abelian integrals on C, and
- p-adic Schneider heights on C.

§1.1. Introduction

Question: Do you know why we care about p-adic integrals and heights?

Question: Why do we care about p-adic Schneider heights?

Answer: It is particularly important because the corresponding p-adic regulator fits into p-adic versions of BSD conjecture.

Some notation:

$(K, \lvert\cdot\rvert)$	-	a finite extension of $\mathbb{Q}_p$ with $p > 2$
$(\mathcal{O}_K, \pi, k)$	-	valuation ring, uniformizer, residue field of K
nice	-	smooth, projective, geometrically integral

§1.2. Overview

§2.1. Mumford Curves

Any subgroup W of $\mathrm{PGL}_2(K)$ acts on $\mathbb{P}^1(\mathbb{C}_p)$. Set

$$\Omega := \Omega_W := \{\textbf{ordinary} \text{ points of } W\} \subset \mathbb{P}^1(\mathbb{C}_p).$$

Definition: A **(p-adic) Schottky group** is a discrete, finitely generated, torsion-free subgroup of $\mathrm{PGL}_2(K)$.

Theorem (Mumford, 1972): Let W be a Schottky group of rank g. Then there is a nice curve C_W of genus g such that

$$C_W^{\mathrm{an}} \simeq \Omega_W / W.$$

Example: Take $q \in K^\times$ with $|q| < 1$, and let W be the cyclic subgroup of $\mathrm{PGL}_2(K)$ generated by $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Then, $C_W \simeq E_q$ (Tate elliptic curve).

§2.1. Mumford Curves

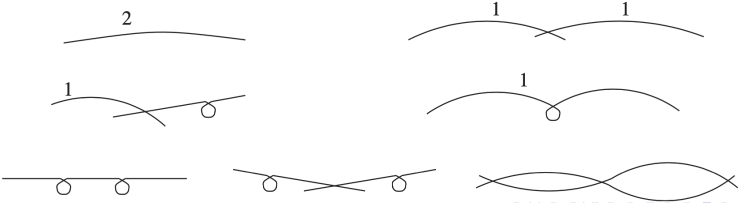
Remark: For any Schottky group W, C_W has *split degenerate* reduction: it has a semistable $\mathcal{O}_K$-model $\mathfrak{X}$ s.t.

- all irreducible components of $\mathfrak{X}_k$ are isomorphic to $\mathbb{P}^1_k$, and
- all double points are k-rational with two k-rational branches.

Example 1: If C_W has genus 1, then

$$\text{split degenerate reduction} = \text{split multiplicative reduction.}$$

Example 2: A genus 2 curve has split degenerate reduction precisely when the special fiber of its stable model is one of the three pictures at the bottom (picture taken from Liu's **Algebraic Geometry and Arithmetic Curves** book):



§2.1. Mumford Curves

Theorem (Mumford, 1972): The map $W \mapsto C_W$ induces a bijection

$$\left\{\begin{matrix}\text{conjugacy classes of Schottky} \\ \text{groups in } \mathrm{PGL}_2(K)\end{matrix}\right\} \to \left\{\begin{matrix}\text{isom. classes of nice curves over} \\ K \text{ with split degenerate reduction}\end{matrix}\right\}.$$

Definition: A curve C over K is called a **Mumford curve** if

$$C \simeq C_W$$

for some Schottky group W in $\mathrm{PGL}_2(K)$.

Remark: Mumford curves = curves with split degenerate reduction.

§2.2. Jacobians of Mumford curves

Now let A be an abelian variety over K of dimension g. We say A is **uniformizable** if

$$A^{\text{an}} \simeq (\mathbb{G}_m^{\text{an}})^g / Q$$

for some lattice Q.

Not every abelian variety is uniformizable.

Question: Which abelian varieties are uniformizable?

Theorem (Mumford, 1972): If A is the Jacobian variety of a Mumford curve over K, then it is uniformizable.

§2.3. Theta functions and period matrices

C - a Mumford curve over K of genus g
W - a Schottky group such that $C^{an} \simeq \Omega/W$

Def'n & fact: For fixed parameters $a, b \in \Omega$, the **theta function** on Ω,

$$\Theta(a, b; z) := \prod_{w \in W} \frac{z - wa}{z - wb}$$

is a meromorphic function and an automorphic form.

Masdeu–Xarles: A polynomial time algorithm to compute Θ functions!

Fact & def'n: There exists a lattice Q_W in $(\mathbb{G}_m^{\mathrm{an}})^g$ which can be defined in terms of Θ such that

$$\mathrm{Jac}(C)^{\mathrm{an}} \simeq (\mathbb{G}_m^{\mathrm{an}})^g / Q_W.$$

It's called the **period matrix** of W.

§3.1. p-adic Abelian integrals

C	-	a nice curve over $\mathbb{Q}_p$ of genus $g \geq 1$
ω	-	a holomorphic 1-form on C
x, y	-	two points on C

To this data, Zarhin (and Colmez, and Vologodsky) associated an integral ${}^{\mathrm{Ab}}\!\int_x^y \omega$, called the **abelian integral**. When C has good reduction, abelian integral = Coleman integral.

There are algorithms to compute these:

- Vologodsky integrals on hyperelliptic curves (Katz–K., K.),
- Vologodsky integrals on “schön” curves (Katz–K. (in progress)).

Assume C is a Mumford curve. Hence $C^{\mathrm{an}} \simeq \Omega/W$ for some Schottky group $W = \langle w_1, \ldots, w_g \rangle$. For $u_{w_i}(z) = \Theta(a, w_i a; z)$, set

$$\eta_i = \frac{u'_{w_i}(z)}{u_{w_i}(z)} dz \quad \Longrightarrow \quad \langle \eta_1, \ldots, \eta_g \rangle = H^0(C, \Omega^1_C).$$

§3.1. p-adic Abelian integrals

Theorem (K.–Masdeu–Müller)

Choose preimages x', y' of x, y in Ω. Then

$$\begin{pmatrix} {}^{\mathrm{Ab}}\!\int_x^y \eta_1 \\ \vdots \\ {}^{\mathrm{Ab}}\!\int_x^y \eta_g \end{pmatrix} = \begin{pmatrix} \log\left(u_{w_1}(D')\right) \\ \vdots \\ \log\left(u_{w_g}(D')\right) \end{pmatrix} - \mathcal{L} \cdot \begin{pmatrix} \operatorname{ord}\left(u_{w_1}(D')\right) \\ \vdots \\ \operatorname{ord}\left(u_{w_g}(D')\right) \end{pmatrix}$$

where $D' = (y') - (x')$, and

$$\mathcal{L} = \log(Q_W) \cdot \operatorname{ord}(Q_W)^{-1} \in K^{g \times g}$$

is the "$\mathcal{L}$-invariant".

Result: To make this formula explicit, there are three main steps:

- determining a Schottky group W uniformizing C,
- determining the period matrix Q_W,
- lifting points from $C^{an} \simeq \Omega/W$ to Ω.

§3.2. p-adic Schneider heights

C - a nice genus curve over $\mathbb{Q}$ of genus $g \geq 1$

Schneider's p-adic height pairing

$$(\cdot,\cdot)_{\mathrm{Sch}} : \mathrm{Div}^0(C) \times \mathrm{Div}^0(C) \to \mathbb{Q}_p,$$

decomposes into local factors:

Theorem (Schneider, 1982): For each prime q, there is a local pairing

$$(\cdot,\cdot)_q : \mathrm{Div}^0(C_q) \times \mathrm{Div}^0(C_q) \to \mathbb{Q}_p$$

such that, for all $D, E \in \mathrm{Div}^0(C)$, we have

$$(D,E)_{\mathrm{Sch}} = \sum_q (D,E)_q = (D,E)_p + \sum_{q \neq p} (D,E)_q.$$

Local components away from p: When $q \neq p$, $(D,E)_q$ is described using arithmetic intersection theory, and it can be computed using van Bommel–Holmes–Müller's work.

Local components at p: This is harder to define. A formula for it was given by Werner in the case where C_p is a Mumford curve:

Theorem (Werner, 1996): Assume

- $C_p^{\mathrm{an}} \simeq \Omega/W$ for some Schottky group W, and
- $D = (x) - (y)$ and $E = (z) - (w)$ for some $x, y, z, w \in C_p$.

Choose preimages x', y', z', w' in Ω. Then

$$(D, E)_p = \log\left(\frac{\Theta(x', y'; z')}{\Theta(x', y'; w')}\right) - \begin{array}{c}\text{another function} \\ \text{in terms of } \Theta.\end{array}$$

Result: To make this formula explicit, there are two main steps:

- determining a Schottky group W uniformizing C,
- lifting points from $C_p^{\mathrm{an}} \simeq \Omega/W$ to Ω.

§4.1. Hyperelliptic Mumford curves

Definition: A matrix $\gamma \in \mathrm{PGL}_2(K)$ is called **elliptic** if its eigenvalues are different but have the same absolute value.

Let $s_0, \ldots, s_g$ be elliptic matrices of order 2 s.t. $\Gamma := \langle s_0, \ldots, s_g \rangle$

- is discrete, and
- equals to the free product $\langle s_0 \rangle * \cdots * \langle s_g \rangle$.

Note that Γ is not a Schottky group. Consider the homomorphism

$$\Gamma \to \{\pm 1\}, \quad s_i \mapsto -1 \text{ for all } i,$$

and set $W :=$ kernel. Then

- W is a Schottky group, freely generated by $w_i := s_i s_0$, $i = 1, \ldots, g$.
- W and Γ have the same set of ordinary points, call it Ω.
- The following map has degree 2:

$$\Omega/W \to \Omega/\Gamma, \quad aW \mapsto a\Gamma;$$

so the Mumford curve $C_W = \Omega/W$ is a double cover of Ω/Γ.

§4.1. Hyperelliptic Mumford curves

For suitably chosen $a, b \in \Omega$, the theta function

$$F(z) := F_{a,b}(z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma b}, \quad z \in \Omega$$

is Γ-invariant and induces an isomorphism $\Omega/\Gamma \simeq \mathbb{P}^{1,\mathrm{an}}$.

As a result, the Mumford curve $C_W = \Omega/W$ is hyperelliptic! The group W is called a **(p-adic) Whittaker** group.

Theorem (van der Put, 1979): Write the fixed points of s_i as $\{a_i, b_i\}$. An equation of C_W is given by

$$y^2 = \prod_{i=0}^{g} (x - F(a_i))(x - F(b_i)).$$

Remark: Every hyperelliptic Mumford curve can be parametrized by a Whittaker group in this way.

§4.1. Hyperelliptic Mumford curves - our setting

We fix the following for the rest of the talk:

C - a hyperelliptic Mumford curve over K

Then there should be

W	-	a Whittaker group such that $C^{an} \simeq \Omega/W$
Γ	-	a group containing W as an index-2 subgroup
$\{w_i\}_i, \{s_j\}_j$	-	generators for W and Γ respectively s.t. $w_i = s_i s_0$
a_i, b_i	-	the fixed points of s_i
S	-	$\{\mathrm{a}_0, b_0, a_1, b_1, \ldots, a_g, b_g\}$
R	-	{roots of the defining polynomial of C}

We may assume

$$S = \{0, b_0, a_1, b_1, \ldots, 1, \infty\}, \quad |a_i|, |b_i| < 1$$
$$R = \{0, r_0, r_1, \ldots r_{2g-2}, 1, \infty\}, \quad |r_i| < 1$$

§4.2. Computing p-adic uniformization

Recall that

$$S = F^{-1}(R), \quad F(z) = F_{a,b}(z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma b} \text{ for suitable } a, b \in \Omega.$$

It suffices to compute the inverse image of R under F. But F depends on Γ, which we don't know yet... We'll simultaneously approximate S and F.

Here is a generalization of Kadziela's approximation theorem:

Theorem (K.–Masdeu–Müller–van der Put)

For $z \in S \setminus \{0, 1, \infty\}$, we have

- $F(z) \equiv 0 \text{ mod } \pi$,
- $F(z) \equiv \begin{cases} -4b_0 \prod_{i=1}^{g-1} \left(1 - \left(\frac{a_i - b_i}{a_i + b_i}\right)^2\right) \text{ mod } \pi^2 & \text{if } z = b_0, \\ -2z \prod_{i=1}^{g-1} \left(1 + \frac{(a_i - b_i)^2}{(a_i + b_i)(2z - a_i - b_i)}\right) \text{ mod } \pi^2 & \text{otherwise}, \end{cases}$
- $F(z) \text{ mod } \pi^t = \prod_{m=0}^{t-2} F_m(z \text{ mod } \pi^t)$ *for* $t \geq 3$.

§4.2. Computing p-adic uniformization

Theorem (K.–Masdeu–Müller–van der Put)

For $z \in S \setminus \{0,1,\infty\}$, we have

- $F(z) \equiv 0 \ mod \ \pi$,
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- $F(z) \ mod \ \pi^t = \prod_{m=0}^{t-2} F_m(z \ mod \ \pi^t)$ *for* $t \geq 3$.

If we know correctly the elements z in $S \setminus \{0,1,\infty\}$ mod π^t, and use them to approximate elliptic matrices $s_i \rightsquigarrow$ group $\Gamma \rightsquigarrow$ theta function $F(z)$, then $F(z)$'s will also correctly approximate the points in R mod π^t.

Summary: We guess the elements in S digit by digit using the approximation thm. This is a brute force algorithm but works quite well!

§4.3. Lifting points from $C^{\mathrm{an}} \simeq \Omega/W$ to Ω

Take $P = (x, y)$ in $C^{\mathrm{an}} \simeq \Omega/W$. Our goal is to compute a lift z of P in Ω.

Consider the commutative diagram

$$\begin{array}{ccc} \Omega & & \\ \downarrow & \searrow & \\ \Omega/W & \rightarrow & \Omega/\Gamma \\ \downarrow & & \downarrow \\ C^{\mathrm{an}} & \rightarrow & \mathbb{P}^{1,\mathrm{an}} \end{array}$$

where the isomorphism $\Omega/\Gamma \simeq \mathbb{P}^{1,\mathrm{an}}$ is induced by $F = F_{a,b} : \Omega \to \mathbb{P}^{1,\mathrm{an}}$.

Using Newton iteration, we can find a $z \in \Omega$ such that $F(z) = x$. Then,

$$\text{the image of } z \text{ in } C \in \{(x, y), (x, -y)\}.$$

Question: But... How do we distinguish?

§4.3. Lifting points from $C^{\mathrm{an}} \simeq \Omega/W$ to Ω

Theorem (K.–Masdeu–Müller–van der Put)

Set $w := w_1 \cdots w_g$, and

$$H(z) := \Theta(a, w(a); z) \cdot \prod_{i=0}^{g} \Theta(a_i, b; z) \cdot \Theta(b_i, s_0(b); z), \quad z \in \Omega.$$

Then

- *The function H is W-invariant, but not Γ-invariant.*
- *Let H also denote the induced element in the function field of $C^{\mathrm{an}} \simeq \Omega/W$. Then*

$$H^2 = \prod_{i=0}^{g} (x - F(a_i))(x - F(b_i)).$$

- *The curve $C^{\mathrm{an}} \simeq \Omega/W$ is parametrized by $z \in \Omega \mapsto (F(z), H(z))$.*

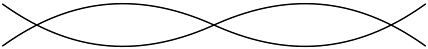
Corollary: If $H(z) = y$, then z is a lift of P. Else $s_0(z)$ is a lift of P.

§5.1. p-adic Abelian integrals; a genus-2 example

Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = (x^2 - x - 1)(x^4 + x^3 - 6x^2 + 5x - 5).$$

It is a Mumford curve over $\mathbb{Q}_5$ and the stable reduction is as follows:



We have $\mathrm{Jac}(C)(\mathbb{Q}) \simeq \mathbb{Z}/12\mathbb{Z}$. Hence, Abelian integrals between any two points $P, Q \in C(\mathbb{Q})$ must vanish.

Goal

Confirm this for points $P = (1, 2)$, $Q = (1, -2) \in C(\mathbb{Q})$.

We will work within the field $K = \mathbb{Q}_5(\pi)$ where π satisfies $\pi^4 = 5$.

§5.1. *p*-adic Abelian integrals; a genus-2 example

The nontrivial fixed points are

$$b_0 = \pi^4 \cdot (14721898770402054636 7 + 76517089843750000000 \cdot \pi^2) + O(\pi^{120}),$$
$$a_1 = \pi^2 \cdot (27018180054400647611 7 + 67760916644468383392 1 \cdot \pi^2) + O(\pi^{120}),$$
$$b_1 = \pi^2 \cdot (86616462826566873383 2 + 65386128612191290661 2 \cdot \pi^2) + O(\pi^{120})$$

and that the matrices

$$w_1 = \begin{pmatrix} \pi^{-8} \cdot (566260181145554 + 507354736328125 \cdot \pi^2) & \pi^{-4} \cdot (28220976712906 + 328063964843750 \cdot \pi^2) \\ \pi^{-8} \cdot (1439946765189457 + 2227783203125000 \cdot \pi^2) & \pi^{-4} \cdot (14110488356453 + 164031982421875 \cdot \pi^2) \end{pmatrix} + O(\pi^{80})$$

$$w_2 = \begin{pmatrix} \pi^{-4} \cdot (384303682828926 + 241352744313936 \cdot \pi^2) & 92591457234474 + 16807499086906 \cdot \pi^2 \\ \pi^{-6} \cdot (154765152323318 + 1334338488576612 \cdot \pi^2) & \pi^{-2} \cdot (23876747388397 + 257490127521949 \cdot \pi^2) \end{pmatrix} + O(\pi^{80})$$

generate a Whittaker group W uniformizing $C_{\mathbb{Q}_5}$. The corresponding period matrix is

$$Q_W = \begin{pmatrix} 5^2 \cdot 23054690584 + O(5^{17}) & 5 \cdot 6733107769 + O(5^{16}) \\ 5 \cdot 6733107769 + O(5^{16}) & 5^2 \cdot 11569554284 + O(5^{17}) \end{pmatrix}.$$

§5.1. p-adic Abelian integrals; a genus-2 example

Let $D = (Q) - (P)$. Its lift is

$$D' = \left(\pi^4 \cdot (161517344335908068786 + 84879185053106998144 \cdot \pi^2)\right) - \left(\pi^2 \cdot (205078598150168020669 + 147110824295460735763 \cdot \pi^2)\right) + O(\pi^{120}).$$

Then

$$(u_{w_1}(D'), u_{w_2}(D')) = \left(5 \cdot 18104738516 + O(5^{16}),\ 5 \cdot 19143541241 + O(5^{16})\right).$$

The $\mathcal{L}$-invariant is

$$\mathcal{L} = \log(Q_W) \cdot \operatorname{ord}(Q_W)^{-1} = \begin{pmatrix} 5^2 \cdot 704651321 & 5 \cdot 6022335363 \\ 5 \cdot 3911365118 & 5^2 \cdot 328336294 \end{pmatrix} + O(5^{15}).$$

Finally, using our formula,

$$\begin{pmatrix} {}^{\mathrm{Ab}}\!\int_P^Q \eta_1 \\ {}^{\mathrm{Ab}}\!\int_P^Q \eta_2 \end{pmatrix} = \begin{pmatrix} \log\left(u_{w_1}(D')\right) \\ \log\left(u_{w_2}(D')\right) \end{pmatrix} - \mathcal{L} \cdot \begin{pmatrix} \operatorname{ord}\left(u_{w_1}(D')\right) \\ \operatorname{ord}\left(u_{w_2}(D')\right) \end{pmatrix}$$

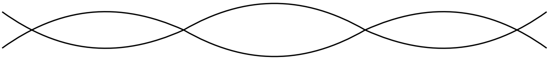
$$= \begin{pmatrix} 5 \cdot 3442076343 + O(5^{15}) \\ 5 \cdot 5553046588 + O(5^{15}) \end{pmatrix} - \mathcal{L} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} O(5^{15}) \\ O(5^{15}) \end{pmatrix}.$$

§5.2. p-adic Schneider heights; a genus-3 example

The modular curve $X_0(39)/\mathbb{Q}$ is a hyperelliptic curve of genus 3 with eq'n

$$y^2 = (x^4 - 7x^3 + 11x^2 - 7x + 1)(x^4 + x^3 - x^2 + x + 1).$$

Over $K = \mathbb{Q}_3(i, \pi)$ where $i^2 = -1$ and $\pi^2 = 3$, it becomes a Mumford curve and its stable reduction is as follows:



Set

$$D = (0, 1) - (\infty_+), \quad E = (0, -1) - (\infty_-).$$

Then D represents a point $P \in J_0(39)(\mathbb{Q})$ of finite order and E represents $-P$. Therefore, the global 3-adic height pairing $(D, E)_{\mathrm{Sch}}$ must vanish.

van Bommel–Holmes–Müller: $\sum_{q \neq 3} (D, E)_q = 0$.

Goal

Compute $(D, E)_3$.

The nontrivial fixed points are

$$
\begin{aligned}
b_0 &= \pi^3 \cdot (16956896945008283367 5\cdot i + 3 \cdot 81345423724756740290 \cdot \pi) + O(\pi^{90}),\\
a_1 &= \pi \cdot (18176551720254840763 60\cdot i + 66994457384064506431 1 \cdot \pi) + O(\pi^{90}),\\
b_1 &= \pi \cdot (13552950269707130347 18 \cdot i + 43399474134979007954 9 \cdot \pi) + O(\pi^{90}),\\
a_2 &= \pi \cdot (18181918218685833971 24 \cdot i + 54847714091525321542 7 \cdot \pi) + O(\pi^{90}),\\
b_2 &= \pi \cdot (24116480122358077625 18 \cdot i + 95321642625024882380 \cdot \pi) + O(\pi^{90}).
\end{aligned}
$$

and that the matrices

$$
w_1 = \begin{pmatrix} \pi^3 \cdot (52793533972127 \cdot i + 3 \cdot 3103846656698 \cdot \pi) & \pi^3 \cdot (31673686785512 \cdot i + 3 \cdot 1417904171591 \cdot \pi) \\ 2 & 83803859730842 + 3 \cdot 52793533972127 \cdot i \cdot \pi \end{pmatrix} + O(\pi^{60}),
$$

$$
w_2 = \begin{pmatrix} \pi \cdot (52144998882143 \cdot i + 66988365732322 \cdot \pi) & \pi \cdot (22184621499875 \cdot i + 30014311084481 \cdot \pi) \\ 2 & 200965097196962 + 52144998882143 \cdot i \cdot \pi \end{pmatrix} + O(\pi^{60}),
$$

$$
w_3 = \begin{pmatrix} \pi \cdot (62883996424702 \cdot i + 67395973180846 \cdot \pi) & \pi \cdot (11576205771028 \cdot i + 68215330456187 \cdot \pi) \\ 2 & 202187919542534 + 62883996424702 \cdot i \cdot \pi \end{pmatrix} + O(\pi^{60}).
$$

generate a Whittaker group W uniformizing $X_0(39)_{\mathbb{Q}_3}$.

Moreover, the corresponding period matrix is

$$Q_W = \begin{pmatrix} 3^3 \cdot 2815951877525 & 3 \cdot 50001091700867 & 3 \cdot 646506176903 \\ 3 \cdot 50001091700867 & 3^4 \cdot 1995748201645 & 3 \cdot 50001091700867 \\ 3 \cdot 646506176903 & 3 \cdot 50001091700867 & 3^3 \cdot 2815951877525 \end{pmatrix} + O(3^{30}),$$

and the lifts of the divisors D, E respectively are

$$\begin{aligned} D' &= \Big(66967506124175 + 158782449129450 \cdot i + (187941388166341 + 96519270558011 \cdot i) \cdot \pi\Big) \\ &\quad - \Big(\pi \cdot (187941388166341 + 109371861536638 \cdot i + (46307875323492 + 52927483043150 \cdot i) \cdot \pi)\Big) + O(\pi^{60}), \\ E' &= \Big(\pi \cdot (17949743928308 + 109371861536638 \cdot i + (46307875323492 + 15702894321733 \cdot i) \cdot \pi)\Big) \\ &\quad - \Big(66967506124175 + 47108682965199 \cdot i + (17949743928308 + 96519270558011 \cdot i) \cdot \pi\Big) + O(\pi^{60}). \end{aligned}$$

Finally, using Werner's formula,

$$\langle D, E \rangle_3 = O(3^{30}).$$

Danke schön!

- *Non-archimedean uniformization and monodromy pairing* - Papikian
- *Schottky groups and Mumford curves* - Gerritzen–van der Put
- *Rigid geometry of curves and their Jacobians* - Lütkebohmert

- *Rigid analytic uniformization of hyperelliptic curves* - Kadziela
- *Algorithms for Mumford curves* - Morrison–Ren
- *Efficient computation of non-archimedean theta functions* - Masdeu–Xarles

- *p-adic abelian integrals and commutative lie groups* - Zarhin
- *Hodge structure on the fundamental group and its application to p-adic integration* - Vologodsky

- *p-adic height pairings I* - Schneider
- *Local heights on Mumford curves* - Werner

- *p-adic integration on bad reduction hyperelliptic curves* - Katz–K.
- *Explicit Vologodsky integration for hyperelliptic curves* - K.
- *Algorithms for hyperelliptic Mumford curves: p-adic uniformization, p-adic integrals and p-adic heights* - K.–Masdeu–Müller–van der Put