Rational points on $X_{\mathrm{ns}}^{+}(19)$

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Part I

Non-split Cartan modular curves

Serre's Uniformity Problem

Let $E/\mathbb{Q}$ be an elliptic curve, $N > 1$ prime and $K/\mathbb{Q}$ Galois such that $E[N] \subset E(K)$. Then $\mathrm{Gal}(K/\mathbb{Q})$ acts on $E[N]$ via

$$(x, y)^\sigma := (\sigma(x), \sigma(y))\,.$$

$\rightsquigarrow$ mod N-Galois representation

$$\bar{\rho}_{E,N} : G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(E[N]) \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\,.$$

Serre's uniformity problem: Is $\bar{\rho}_{E,N}$ surjective for every non-CM elliptic curve $E/\mathbb{Q}$ and every prime $N >$ some constant N_0?

Conjecture (Sutherland, Zywina). Yes, and $N_0 = 37$.

Let $G \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \rightsquigarrow$ modular curve $X_G := X(N)/G$ such that

$$E/\mathbb{Q} \text{ with } \mathrm{im}(\bar{\rho}_{E,N}) \subset G \rightsquigarrow \text{ non-cuspidal } P \in X_G(\mathbb{Q})\,.$$

Rational points on modular curve

Idea. Find $X_G(\mathbb{Q})$ for all maximal proper subgroups $G \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. These are

(1) Borel ✓Mazur, 1977, $N > 37$

(2) Exceptional ✓Serre, 1972, $N > 13$; $N = 13$?

(3) Normalizers of split Cartan ✓Bilu–Parent–Rebolledo, 2013, $N > 13$; $N = 13$?

(4) Normalizers of non-split Cartan ?

Write $X_{\mathrm{s}}^+, X_{\mathrm{ns}}^+$ for X_G as in (3) and (4).

Balakrishnan–Dogra–M.–Tuitman–Vonk, 2019, 2023: Using skimmed quadratic Chabauty, computed rational points on

$$X_{\mathrm{s}}^+(13) \simeq X_{\mathrm{ns}}^+(13),\ X_{S_4}(13),\ X_{\mathrm{ns}}^+(17)$$

with respective genera $g = 3, 3, 6$,

Non-split Cartan

Rational points on $X_{\mathrm{ns}}^{+}(N)$ are also interesting for composite N, e.g. for Mazur's Program B. Some recent examples:

- Furio–Lombardo, 2025: $X_{\mathrm{ns}}^{+}(7^2)$, $g = 69$ (!) "following" Poonen–Schaefer–Stoll. See talk of Zureick–Brown.
- Balakrishnan–Betts–Hast–Jha–M., 2024: $X_{\mathrm{ns}}^{+}(27)$, $g = 12$ using skimmed quadratic Chabauty over number fields.

Some open cases:

- $X_{\mathrm{ns}}^{+}(5^2)$: $g = 14$
- $X_{\mathrm{ns}}^{+}(11^2)$: $g = 511$
- $X_{\mathrm{ns}}^{+}(N)$, $N > 17$ prime,
  $g = \frac{1}{24}\left(N^2 - 10N + 23 + 6\left(\frac{-1}{N}\right) + 4\left(\frac{-3}{N}\right)\right)$

This talk: $N = 19$.

Part II

(Skimmed) Quadratic Chabauty

Kim's program

- $X/\mathbb{Q}$: nice curve of genus $g > 1$
- p: a prime of good reduction for X.

Kim (2006, 2009): conjectural extension of Chabauty-Coleman using unipotent fundamental groups and p-adic Hodge theory.

Quadratic Chabauty (QC): depth 2, i.e. find $X(\mathbb{Q}_p)_U$, where

$$X(\mathbb{Q}_p) \supset X(\mathbb{Q}_p)_1 \supsetneq X(\mathbb{Q}_p)_U \supset X(\mathbb{Q}_p)_2 \supset \ldots \supset X(\mathbb{Q}).$$

Skimmed QC (Balakrishnan–Dogra, 2017):

- U: maximal quotient of U_2 such that $[U, U] \simeq \mathbb{Q}_p(1)^{\oplus n}$
- Balakrishnan–Dogra used Nekovář's p-adic heights
- Alternative approaches
  - Poincaré torsors (Edixhoven–Lido, 2023)
  - p-adic Arakelov theory (Besser–M.–Srinivasan, 2025)

Balakrishnan–Dogra, Dogra: semi-skimmed and full-fat QC.

Setup for skimmed quadratic Chabauty

For simplicity, we now assume

(1) $r := \mathrm{rk}(J(\mathbb{Q})) = g$, $J = \mathrm{Jac}(X)$

(2) $\mathrm{rk}\,\mathrm{NS}(J) > 1$, where $\mathrm{NS} = \mathrm{Pic}\,/\,\mathrm{Pic}^0$

(3) $\log\colon J(\mathbb{Q}) \otimes \mathbb{Q}_p \cong \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee$

Chabauty–Coleman uses linear relations in $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee$ satisfied by the image of $J(\mathbb{Q})$ under log.

Idea. Replace linear relations by quadratic relations.

p-adic heights

The p-adic height pairing

$$h\colon J(\mathbb{Q}) \times J(\mathbb{Q}) \to \mathbb{Q}_p$$

is a symmetric bilinear pairing, similar to Néron–Tate heights.

Our assumption

$$\log\colon J(\mathbb{Q}) \otimes \mathbb{Q}_p \cong \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee$$

implies that we can write h in terms of a basis of bilinear pairings on $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee$ by evaluating in enough points.

We get a locally analytic extension of h to $\mathbb{Q}_p$-points.

p-adic heights

Want to pull

$$h\colon J(\mathbb{Q}) \times J(\mathbb{Q}) \to \mathbb{Q}_p$$

back to $X(\mathbb{Q})$.

Fix $b \in X(\mathbb{Q})$ and $Z \in \ker(\mathrm{NS}(J) \to \mathrm{NS}(X)) \setminus \{0\}$.

$\rightsquigarrow$ $E_Z \in \mathrm{End}(J)$

$\rightsquigarrow$ $c_Z \in J(\mathbb{Q})$: Chow–Heegner point.

For $P \in X(\mathbb{Q})$,

$$h_Z(P) := h([P-b], E_Z([P-b]) + c_Z)$$

and h_Z extends to a locally analytic function

$$h_Z\colon X(\mathbb{Q}_p) \to \mathbb{Q}_p\ .$$

Skimmed quadratic Chabauty: Theory

$$h_Z(P) := h([P-b], E_Z([P-b]) + c_Z)\ , P \in X(\mathbb{Q})$$

Theorem (Balakrishnan–Dogra, 2017).

There is a local decomposition

$$h_Z(P) = \sum_{q \text{ prime}} h_{Z,q}(P)\,, \quad \text{where} \quad h_{Z,q} \colon X(\mathbb{Q}_q) \to \mathbb{Q}_p\,.$$

Moreover

(1) $h_{Z,p}$ is locally analytic;

(2) for $q \neq p$, $h_{Z,q}(X(\mathbb{Q}_q))$ is finite. For potentially good q, $h_{Z,q} \equiv 0$;

(3) $F := h_Z - h_{Z,p} \colon X(\mathbb{Q}_p) \to \mathbb{Q}_p$ has finite fibers.

Skimmed quadratic Chabauty: algorithm

$$F := h_Z - h_{Z,p} \colon X(\mathbb{Q}_p) \to \mathbb{Q}_p \,.$$

For $P \in X(\mathbb{Q})$,

$$F(P) = \sum_{\text{all } q} h_{Z,q}(P) - h_{Z,p}(P) = \sum_{q \neq p} h_{Z,q}(P) \,.$$

Assume $h_{Z,q} \equiv 0$ for all $q \neq p$, then $F(X(\mathbb{Q})) = 0$.

Can compute $X(\mathbb{Q})$ if we can explicitly

(a) find Z such that $h_{Z,q}(X(\mathbb{Q}_q)) \equiv 0$ for all $q \neq p$; [1]

(b) expand $h_{Z,p}$ on $X(\mathbb{Q}_p)$; (this talk for $X_{\mathrm{ns}}^{+}(19)$)

(c) expand h_Z on $X(\mathbb{Q}_p)$; (standard for $X(\mathbb{Q})$ "sufficiently big")

(d) find roots of F with provable precision (standard).

Implemented in the Magma-function `QCModAffine`,
on `https://github.com/steffenmueller/QCMod`.

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[1] Or find all values of $h_{Z,q}(X(\mathbb{Q}_q))$

Local heights away from p

Finding $h_{Z,q}(X(\mathbb{Q}_q))$ for all $q \neq p$ is hard in general, see talks of

- Duque–Rosero
- Vonk
- Derickx, Adžaga
- Besser

Geometric QC doesn't have this problem.

Part III

Local heights at p

Local heights at p

Balakrishnan–Dogra: $h_{Z,p}(P)$ is the Nekovář height $h_{Z,p}(M(P))$. Here

- $M(P) \simeq P^*\mathcal{A}_Z$ is a filtered ϕ-module, where $\mathcal{A}_Z$ is a unipotent isocrystal with graded pieces

$$\mathbb{Q}_p,\ \mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})^\vee,\ D_{\mathrm{cris}}(\mathbb{Q}_p(1)) \quad \text{recalled below;}$$

- $h_{Z,p}(M(P))$ can be computed from Hodge filtration and Frobenius structure on $\mathcal{A}_Z$. Via Besser's theory, this is equivalent to computing certain Coleman integrals.

Hodge filtration

Choose an affine open $Y/\mathbb{Q}$ in X, let $d = \#(Y \setminus X)(\bar{\mathbb{Q}})$.

Fix $\omega_1, \ldots, \omega_{2g+d-1} \in H^0(Y_{\mathbb{Q}}, \Omega^1)$, generating $H^1_{\mathrm{dR}}(Y_{\mathbb{Q}})$, such that the classes of $\vec{\omega} = (\omega_1, \ldots, \omega_{2g})$ form a basis of $H^1_{\mathrm{dR}}(X_{\mathbb{Q}}) \supset \mathrm{H}^0(X_{\mathbb{Q}}, \Omega^1)$.

On Y we can trivialize

$$\mathcal{A}_Z|_Y \simeq (\mathbb{Q}_p \oplus \mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})^\vee \oplus D_{\mathrm{cris}}(\mathbb{Q}_p(1))) \otimes \mathcal{O}_Y$$

such that the connection on $\mathcal{A}_Z|Y$ is

$$\nabla = d + \Lambda, \quad \text{where } \Lambda = -\begin{pmatrix} 0 & 0 & 0 \\ \vec{\omega} & 0 & 0 \\ \eta & \vec{\omega}^t Z & 0 \end{pmatrix}$$

where η is logarithmic, and ∇ extends to a holomorphic connection on X. This essentially determines the Hodge filtration on $\mathcal{A}_Z$.

Computing the Hodge filtration

Hardest computational steps:

- Compute local coordinates t_P at each $P \in X \setminus Y$.
- At each $P \in X \setminus Y$, expand all ω_i into Laurent series;
- linear algebra in terms of these expansions.

Original algorithm worked over splitting field of $X \setminus Y$, but can work over individual $\mathbb{Q}(P)$ (see also Balakrishnan–Betts–Hast–Jha–M.)

Still, for large genus this is the bottleneck of `QCModAffine`.

Takeaway: Want arithmetic in these fields to be fast!

But: Also need "enough" points in $Y(\mathbb{Q})$ to extend h_Z.

Frobenius structure

By a crystalline comparison theorem of Chiarellotto-Le Stum, $\mathcal{A}_Z$ admits a Frobenius structure $F\colon \mathsf{F}_p^*\mathcal{A}_Z \to \mathcal{A}_Z$, horizontal w.r.t ∇, where F_p is the action of Frobenius on $H^1_{\mathrm{dR}}(X_{\mathbb{Q}_p})$.

Tuitman's algorithm gives a matrix Φ and vector $\vec{f}$ such that

$$\mathsf{F}_p^*(\omega_i) = \sum_{j=1}^{2g} \Phi_{ij}\omega_j + df_i\,.$$

Then the matrix of F is G^{-1}, where

$$G = \begin{pmatrix} 1 & 0 & 0 \\ \vec{f} & \Phi & 0 \\ h & \vec{g}^t & p \end{pmatrix} \qquad \text{such that} \qquad dG = G\Lambda - \mathsf{F}_p^*(\Lambda)G.$$

The differential equation uniquely determines G; we solve it using Tuitman's algorithm.

Tuitman's algorithm

If $Y/\mathbb{Q}$: $Q(x,y)=0$ is a (possibly singular) affine patch of X s.t.

$$x\colon U := Y \setminus \{\text{ramification divisor}\} \to \mathbb{P}^1 \setminus \{\text{branch divisor}\}$$

has a good compactification over $\mathbb{Z}_p$, call (Y,p) a Tuitman pair.

Tuitman's algorithm extends Kedlaya's and computes reductions in

$$H^1_{\mathrm{dR}}(X_{\mathbb{Q}_p}) \subset H^1_{\mathrm{rig}}(U_{\mathbb{F}_p})\,.$$

Runtime:

$$\widetilde{O}(pd_x^4 d_y^2(n^2 + d_x d_y n))\,,$$

where $d_x = \deg_y Q$ and n is the p-adic precision.

Takeaway: Want small d_x and d_y!

Part IV

Quadratic Chabauty for $X_{\mathrm{ns}}^{+}(19)$

Main result

From now on $X = X_{\mathrm{ns}}^{+}(19)$. Last month, we showed:

Theorem. (Balakrishnan–M.–Vonk, June 26): $X_{\mathrm{ns}}^{+}(19)(\mathbb{Q})$ consists precisely of 7 CM-points, with respective discriminants

$$-4, -7, -11, -16, -28, -43, -163\,.$$

Corollary. Complete classification of all 19-adic Galois representations of elliptic curves.

Rest of the talk: Main steps.

Local heights for $X_{\mathrm{ns}}^{+}(19)$

The modular curve $X_{\mathrm{ns}}^{+}(19)$ has genus 8.

The only bad prime is at 19. We can find polynomial expressions Z in the Hecke operator T_{13} such that $h_{Z,19} \equiv 0$.

In practice, we compute T_{13} from the Frobenius matrix via Eichler–Shimura.

A model of $X_{\mathrm{ns}}^{+}(19)$

Mercuri and Schoof computed the canonical embedding:

MODULAR FORMS INVARIANT UNDER NON-SPLIT CARTAN GROUPS 1987

TABLE 8.2. Coefficients of quadrics defining $X_{\mathrm{ns}}^{+}(19)$

−1	1	0	0	−1	0	0	−1	1	1	0	2	0	1	0
1	0	0	0	1	0	0	−1	−2	−1	1	0	3	−2	4
0	−1	−1	0	1	0	1	2	−1	0	1	−3	1	0	1
0	1	−1	0	−1	1	0	1	−2	0	1	2	1	0	0
−1	−1	1	−2	1	0	0	−1	1	−1	0	1	1	2	2
0	0	0	−1	2	0	−1	0	−1	−1	0	−2	0	0	1
0	1	0	−1	0	0	0	0	−1	−1	−1	−1	0	0	−1
0	0	0	0	1	−1	1	−1	0	0	0	1	2	−1	−1
0	0	1	1	0	0	0	0	0	0	−1	0	0	0	1
1	−1	1	0	−1	0	−1	2	1	−1	0	0	−1	0	0
0	−1	1	0	1	0	0	0	−1	1	0	−1	2	−1	0
0	1	−1	0	0	1	0	−1	0	1	0	0	1	−1	−1
0	0	0	0	0	2	−1	0	1	2	−1	0	2	−1	−1
1	0	−1	−1	−1	1	1	1	−1	0	0	0	0	1	−2
0	0	1	1	0	0	0	0	0	−1	1	1	0	1	−1
0	−2	0	0	0	0	0	2	0	0	−1	−1	−1	0	0
0	1	1	0	−1	−1	0	−2	2	−1	0	0	0	1	−1
1	−1	0	−2	1	−1	0	1	1	1	2	−1	0	0	−1
1	−1	1	−1	0	−1	−1	−2	2	0	−1	0	0	0	−1
1	−1	0	−1	−1	1	0	1	−1	0	−1	0	0	0	0
1	0	1	0	−2	2	−2	1	−1	0	0	−1	0	−1	0
1	0	1	0	0	1	0	1	−2	−1	1	1	1	−1	1
0	0	−1	0	−1	0	0	−1	0	−1	0	2	−1	−1	−3
1	−1	1	0	0	1	0	0	0	0	−1	0	0	−3	0
−1	2	0	0	−1	1	0	0	0	1	−1	−1	−2	0	−1
1	0	1	0	1	0	0	−1	0	−1	0	1	1	−1	1
−1	0	−1	−1	0	0	0	−1	1	0	0	0	−1	0	−2
−1	1	−1	0	0	−1	2	−2	0	−1	0	0	−2	−1	−1
0	−2	1	−1	1	−2	−1	1	0	1	−1	−1	−2	−1	−1
−1	0	0	1	1	−1	−1	0	−1	1	1	−1	0	−3	−1
0	0	0	0	0	0	1	0	0	0	0	0	0	−1	0
0	0	0	−1	0	0	0	2	0	1	1	0	−2	0	0
0	0	0	0	1	0	−1	0	0	0	0	−1	−3	−1	1
1	0	0	−1	−1	0	0	0	1	0	1	1	0	0	0
0	0	−1	−1	−2	1	1	1	1	0	−1	0	−1	1	1
0	0	0	0	1	0	−1	0	0	0	0	−1	0	0	2

Here the rows contain the coefficients of the 36 monomials $x_i x_j$ with $1 \leq i \leq j \leq 8$ in lexicographic order. Each column corresponds to the equation of a quadric in $\mathbb{P}^7$.

$X_{\mathrm{ns}}^{+}(19)$ via LMFDB

LMFDB (beta) records data about this curve as `19.171.8.a.1` and has this canonical model:

$$
\begin{cases}
3xy - xz - 2xt + xu - xv - xr - y^2 + yz - yu + yr - z^2 + zt - zu - zv - zr + tu + tv + ur, \\
xy - xu + xr - 2y^2 - 2yz - yu + yv + yr + z^2 + zt + 2zu + t^2 + tv + tr - u^2 - uv - v^2 - vr + r^2, \\
x^2 - xy + xz - xt + xu + 2xr + y^2 + yw + yt + yv - yr - 2z^2 - zw + zu - zv - wr - tu - 2tr - vr, \\
xz - xw + y^2 - yz + 2yt + yv - yr - z^2 + zw - 2zt + zu - zv + zr + wv - wr - tu - tv - 2tr + u^2 + uv + ur, \\
xy - xz - xu - xv - xr + 2yz - yw - yu + yv + yr - z^2 - zv - zr + wu - wr + u^2 + uv + 2ur - r^2, \\
2xy - xz + xv + 3yz + yw - 2yt + yu - yv + yr - z^2 - zw + zt + zv - zr + wu + wv - wr - t^2 - tv + tr + u^2 + vr - r^2, \\
2xy + xw - xt + 3xu - yw - yu - yv - z^2 - zw - zt - 2zu + zv - zr + wu - wv + tu - tv - tr + u^2 - uv + vr, \\
xz + xt + xu + yw + yt + 2yu + yv - z^2 - zw - 2zt - zu + zr - wu - wv - wr - tu - 2tv + u^2, \\
xy - xz + xw - xt + xu - y^2 - 2yw - yu - yv + z^2 + 2zt - zu + zv + wt + 2wr - t^2 + tu + tr + ur + vr + r^2, \\
2xy + xt + y^2 + 2yz - yt - 2z^2 + zt - zv - zr + w^2 - wt + wv - wr - t^2 - tr - r^2, \\
2x^2 - xy - xz - xw + xt + 2xu - 2xv + xr + y^2 + 2yw + yu + yv - z^2 - 2zt + zv + zr + 2wv - wr - 2tv - tr + u^2 - vr - r^2, \\
x^2 + 2xy - 2xz + xw - xt - 2xu - 2xv + xr - y^2 + yu - 2zw + 2zt + 2zu + 2zv + wu - wv - wr - tu + tv + tr - u^2 + v^2 + vr + r^2, \\
xz + 2xt - xu + xr - yz - yw + 2yu - yv - z^2 - zt - zv + wt - wu - t^2 + tu - 2tv + 2uv + 2ur - 2vr, \\
xz - xt - xv + xr + y^2 + yz - yw + 2yu - yr - z^2 + zw - zt + 2zu + zr - 3wt - wu + 2wv - t^2 - 3tu + 2uv + 2v^2 + vr, \\
x^2 + xy - xz + xw + xu - 4xr - y^2 - yw - yt - yv + yr - z^2 + 2zt - zu - zr + 2w^2 - wt + 2wu + wv - 2wr - t^2 + 2tr + u^2 - 2ur + vr
\end{cases}
$$

$X_{\mathrm{ns}}^{+}(19)$ via LMFDB

...as well as a singular plane model of degree 14:

$$\begin{aligned}
Q(x,y,z) := {} & 27136x^{13}y - 27136x^{13}z + 727616x^{12}y^2 - 554624x^{12}yz + 57664x^{12}z^2 + 1861592x^{11}y^3 + 6538568x^{11}y^2z - 2061544x^{11}yz^2 \\
& + 495688x^{11}z^3 + 1703745x^{10}y^4 + 32087096x^{10}y^3z + 26903390x^{10}y^2z^2 - 5191186x^{10}yz^3 + 23427x^{10}z^4 + 829785x^9y^5 \\
& + 41143023x^9y^4z + 153325216x^9y^3z^2 + 48277880x^9y^2z^3 - 36070892x^9yz^4 + 1366823x^9z^5 - 3832445x^8y^6 + 5307489x^8y^5z \\
& + 238089449x^8y^4z^2 + 361378827x^8y^3z^3 - 108372388x^8y^2z^4 - 72863208x^8yz^5 + 12415879x^8z^6 - 9958990x^7y^7 \\
& - 37004640x^7y^6z + 104098943x^7y^5z^2 + 628568688x^7y^4z^3 + 197490422x^7y^3z^4 - 515196951x^7y^2z^5 + 45937550x^7yz^6 + 15818876x^7z^7 \\
& - 5163086x^6y^8 - 46181833x^6y^7z - 69635265x^6y^6z^2 + 361191457x^6y^5z^3 + 704064283x^6y^4z^4 - 722414499x^6y^3z^5 - 586448544x^6y^2z^6 \\
& + 293855244x^6yz^7 - 15576754x^6z^8 + 2908982x^5y^9 - 12273384x^5y^8z - 100636780x^5y^7z^2 - 19876738x^5y^6z^3 + 575823337x^5y^5z^4 - 129951037x^5y^4z^5 \\
& - 1591974838x^5y^3z^6 + 145564182x^5y^2z^7 + 334491130x^5yz^8 - 55044976x^5z^9 + 2464137x^4y^{10} + 11228545x^4y^9z - 27991502x^4y^8z^2 \\
& - 110001881x^4y^7z^3 + 162994303x^4y^6z^4 + 351995724x^4y^5z^5 - 1235908035x^4y^4z^6 - 1110526849x^4y^3z^7 + 779635982x^4y^2z^8 + 122154458x^4yz^9 - 60335583x^4z^{10} \\
& - 30132x^3y^{11} + 3458789x^3y^{10}z + 11421025x^3y^9z^2 - 28654424x^3y^8z^3 - 33183393x^3y^7z^4 + 346437983x^3y^6z^5 - 140387750x^3y^5z^6 \\
& - 1554648562x^3y^4z^7 + 210200627x^3y^3z^8 + 554091566x^3y^2z^9 - 35677174x^3yz^{10} - 38264537x^3z^{11} - 5580x^2y^{12} - 780042x^2y^{11}z \\
& + 525692x^2y^{10}z^2 + 285606x^2y^9z^3 - 16284816x^2y^8z^4 + 89183176x^2y^7z^5 + 282087025x^2y^6z^6 - 435871731x^2y^5z^7 - 747432725x^2y^4z^8 \\
& + 582620902x^2y^3z^9 + 123196247x^2y^2z^{10} - 47793170x^2yz^{11} - 15319795x^2z^{12} - 48360xy^{12}z - 1149488xy^{11}z^2 - 1465131xy^{10}z^3 \\
& - 7320969xy^9z^4 - 7411260xy^8z^5 + 148668689xy^7z^6 + 70263883xy^6z^7 - 467661492xy^5z^8 + 102084435xy^4z^9 + 162540212xy^3z^{10} + 6295308xy^2z^{11} \\
& - 21419232xyz^{12} - 3304024xz^{13} - 104780y^{12}z^2 + 2439662y^{11}z^3 + 1613697y^{10}z^4 - 29689893y^9z^5 - 1697152y^8z^6 + 119398916y^7z^7 \\
& - 51269804y^6z^8 - 146420631y^5z^9 + 115042701y^4z^{10} - 9088099y^3z^{11} + 3839957y^2z^{12} - 4579152yz^{13} - 224124z^{14}
\end{aligned}$$

After earlier failed attempts in 2021, we tried working with (essentially) this model in August 2025, without getting past a symplectic basis of de Rham cohomology.

$X_{\mathrm{ns}}^{+}(19)$ via Karabıyık

Tuitman pairs for curves of genus $4, 5, 6$ are due to Adžaga–Arul–Beneish–Chen–Chidambaram–Keller–Wen (2023), but these didn't help.

Eray Karabıyık kindly computed more than 70 singular plane models for $X_{\mathrm{ns}}^{+}(19)$. The lowest degree he found was degree 11:

$$
\begin{aligned}
Q(x,y) := y^{11} &+ (x+7)y^{10} + (12x^2+24x+27)y^9 \\
&+ (29x^3+131x^2+161x+80)y^8 \\
&+ (40x^4+281x^3+624x^2+542x+177)y^7 \\
&+ (-68x^5-27x^4+673x^3+1409x^2+1044x+274)y^6 \\
&+ (-42x^6-211x^5+64x^4+1425x^3+2283x^2+1405x+310)y^5 \\
&+ (156x^7+601x^6+1135x^5+2032x^4+3232x^3+3071x^2+1454x+267)y^4 \\
&+ (31x^8+495x^7+1822x^6+3539x^5+4797x^4+4779x^3+3135x^2+1149x+175)y^3 \\
&+ (-109x^9-428x^8-331x^7+1096x^6+3338x^5+4643x^4+3982x^3+2129x^2+643x+83)y^2 \\
&+ (-10x^{10}-189x^9-657x^8-806x^7+175x^6+1772x^5+2529x^4+1954x^3+901x^2+233x+26)y \\
&+ 23x^{11}+84x^{10}+50x^9-213x^8-442x^7-240x^6+247x^5+510x^4+398x^3+171x^2+40x+4.
\end{aligned}
$$

Again, `QCModAffine` does not get beyond a symplectic basis.

More models for $X_{\mathrm{ns}}^{+}(19)$

In February, we started to use AI tools to write `Magma` code to produce lower degree models.

- We want to compute a birational projection of $X_{\mathrm{ns}}^{+}(19)$ to a plane curve in $\mathbb{P}^2$.
- A generic linear projection $\pi \colon \mathbb{P}^7 \dashrightarrow \mathbb{P}^2$ is defined by choosing a center of projection Λ, where $\dim \Lambda = 7 - 2 - 1 = 4$.
- Generic Λ will result in high degree and large coefficients.
- To achieve lower degree, choose Λ such that it intersects $X_{\mathrm{ns}}^{+}(19)$ at 5 known $P_1, \ldots, P_5 \in X_{\mathrm{ns}}^{+}(19)(\mathbb{Q})$ in general position.

More models for $X_{\mathrm{ns}}^{+}(19)$, continued

- The projection map π is determined by linear forms $L_1, L_2, L_3 \in \mathbb{Q}[x_1, \ldots, x_8]$.
- For π to project away from Λ, need $L_i(P_j) = 0$ for all i, j.
- We construct a 5×8 matrix M whose rows are the coordinates of $P_1 \ldots, P_5$ and compute the right kernel of M. Any basis of this kernel provides coefficients of possible L_1, L_2, L_3.
- To find the image, use elimination theory via Gröbner bases.
- To avoid coefficient explosion, use lattice basis reduction.
- Work modulo small primes first to rule out candidates.
- Additional conditions: Look for Tuitman pairs with small degree points at infinity.

An affine model that works

Gemini Pro 3.1 and GPT 5.2 Pro produced the Tuitman pair $(Y_1, 13)$, where $Y_1\colon Q_1(x,y) = 0$, and

$$\begin{aligned}6Q_1 = {} & 9x^7y + 66x^6y^2 + 9x^6 + 157x^5y^3 - 69x^5y^2 + 33x^5y + 18x^5 \\ & + 162x^4y^4 - 304x^4y^3 - 3x^4y^2 - 6x^4y - 69x^4 \\ & + 71x^3y^5 - 394x^3y^4 + 23x^3y^3 + 146x^3y^2 - 246x^3y - 72x^3 \\ & - 26x^2y^6 + 34x^2y^5 + 35x^2y^4 + 387x^2y^3 + 107x^2y^2 - 14x^2y + 13x^2 \\ & - 13xy^7 + 83xy^6 - 18xy^5 - 317xy^4 + 8xy^3 + 138xy^2 + 77xy + 10x \\ & + 6y^8 - 38y^7 + 53y^6 + 40y^5 - 77y^4 - 2y^3 + 19y^2 - 1\,.\end{aligned}$$

We successfully ran `QCModAffine` on Y_1 in 13 hours [2].

End of proof, right? Not quite, unfortunately...

[2] on a single core of a 56-core 2.2 GHz Intel Xeon server with 256 GB RAM

QC coverings

`QCModAffine` only finds rational points in good disks of Y_1, where a residue disk in $Y_1(\mathbb{Q}_p)$ is good if it does not reduce to a singular point or a ramification point of $x\colon Y_1 \to \mathbb{P}^1$.

So we need affine patches $Y_2, \ldots, Y_m$ of X over $\mathbb{Q}$ such that

(1) (Y_i, p) is a Tuitman pair for all Y_i

(2) every $P \in X(\mathbb{Q}_p)$ is in a good disk of at least one Y_i.

Call such $(Y_1, \ldots, Y_m; p)$ a QC covering.

New goal: Find QC covering $(Y_1, Y_2; 13)$ for $X_{\mathrm{ns}}^+(19)$!

Idea: Act on Y_1 via GL_2.

Problem: This maps singular points to singular points. :-(

Fortunately, $Y_1/\mathbb{F}_{13}$ has a unique singular point, and its preimages on the canonical model of $X_{\mathrm{ns}}^+(19)$ are not defined over $\mathbb{F}_{13}$. :-)

Second affine patch

Trying Claude and GPT5.5, we obtain a QC covering $(Y_1, Y_2; 13)$, where $Y_2\colon Q_2(x,y) = 0$ has degrees $(1,1,2,2,2)$ at infinity and

$$\begin{aligned}171 Q_2 := {} & 6x^8 + 58x^7y + 13x^7 + 193x^6y^2 + 99x^6y - 26x^6 \\ & + 172x^5y^3 + 114x^5y^2 - 278x^5y - 71x^5 - 417x^4y^4 - 843x^4y^3 \\ & - 1185x^4y^2 - 316x^4y + 162x^4 - 1066x^3y^5 - 2752x^3y^4 - 2133x^3y^3 \\ & + 289x^3y^2 + 992x^3y - 157x^3 - 705x^2y^6 - 2322x^2y^5 - 251x^2y^4 \\ & + 3492x^2y^3 + 2061x^2y^2 - 873x^2y + 66x^2 + 84xy^7 + 747xy^6 \\ & + 3906xy^5 + 6314xy^4 + 1518xy^3 - 1641xy^2 + 264xy - 9x \\ & + 171y^8 + 1220y^7 + 3493y^6 + 3828y^5 + 67y^4 - 1064y^3 + 273y^2 - 18y\,.\end{aligned}$$

`QCModAffine` succesfully finished on Y_2 in 33 hours, proving:

Theorem. (Balakrishnan–M.–Vonk, June 26): $X_{\mathrm{ns}}^+(19)(\mathbb{Q})$ consists precisely of 7 CM-points, with respective discriminants

$$-4, -7, -11, -16, -28, -43, -163\,.$$

What's next?

- $N = 23$? $N = 29$?
- $N = 5^2$?
- Optimize Hodge filtration computation:
  - Figure out dependence on various parameters.
  - Work over $\mathbb{Q}_p$? Needs function fields...
  - Use modular approach of Rendell, Leonhardt-Rendell?
- If $X(\mathbb{Q})$ is too small to solve for the pairing h, need algorithms for construction of h by Coleman–Gross (done for hyperelliptic X; general case in progress (Balakrishnan–Gajović–M.))
- Larger N: Need model-free QC. Ideas:
  - geometric QC: see talks by Lido and Hashimoto
  - N-adic uniformization: see talk by Kaya
- Algorithms for semi-skimmed and full-fat QC, building on work of Dogra and Berry.