Chabauty–Kim for the twice-punctured line

Winter Workshop Chabauty–Kim, Exercise session

Martin Lüdtke Heidelberg, 14 February 2024

The aim of this exercise is to give a Chabauty–Kim proof of the fact that $\mathbb{G}_m(\mathbb{Z}) = \{\pm 1\}$.

Let K be a field of characteristic zero, $\overline{K}/K$ an algebraic closure and $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group.

(a) Show that the morphisms $\mathbb{G}_{m,\overline{K}} \xrightarrow{n} \mathbb{G}_{m,\overline{K}}$, $x \mapsto x^n$ are finite étale covers for all $n \in \mathbb{N}$. Show that every connected finite étale cover of $\mathbb{G}_{m,\overline{K}}$ is of this form.
Hint: reduce to the case $K \subseteq \mathbb{C}$ and use the equivalence between finite étale covers of X and finite topological covers of $X(\mathbb{C})$.

Étale fundamental groups and path spaces can be described via the *universal pro-covering*. The universal pro-covering of X is a pro-object $\widetilde{X} = (X_i)_i$ of $\mathrm{Cov}(X)$ such that for every finite étale cover $Y \to X$ there is a morphism $\widetilde{X} \to Y$ over X. By the definition of morphisms of pro-objects, $\mathrm{Hom}(\widetilde{X}, Y) = \varinjlim_i \mathrm{Hom}(X_i, Y)$. For $b \in X(\overline{K})$ a base point and $\tilde{b} = (b_i)_i \in \varprojlim_i F_b(X_i)$ compatible points in the fibre over b, the pointed pro-universal cover $(\widetilde{X}, \tilde{b})$ pro-represents the fibre functor $F_b\colon \mathrm{Cov}(X) \to \mathrm{FinSet}$, i.e. one has a natural isomorphism

$$F_b(Y) \cong \varinjlim_i \mathrm{Hom}(X_i, Y).$$

for $Y \in \mathrm{Cov}(X)$. From this one obtains

$$\pi_1^{\text{ét}}(X; b, x) = \varprojlim_i F_x(X_i)$$

(b) Show that the covers $(\mathbb{G}_{m,\overline{K}} \xrightarrow{n} \mathbb{G}_{m,\overline{K}})_n$ form a pro-universal cover of $\mathbb{G}_{m,\overline{K}}$. Conclude that for $x \in \mathbb{G}_m(K)$ we have

$$\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}; 1, x) \cong \varprojlim_n x^{1/n},$$

where $x^{1/n}$ denotes the set of n-th roots of x in $\overline{K}$. In particular, the étale fundamental group of $\mathbb{G}_{m,\overline{K}}$ is given by

$$\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1) = \hat{\mathbb{Z}}(1) := \varprojlim_n \mu_n(\overline{K}).$$

Now fix an auxiliary prime p. Taking the $\mathbb{Q}_p$-prounipotent completion, the previous item shows that the ($\mathbb{Q}_p$-points of the) $\mathbb{Q}_p$-pro-unipotent fundamental group $U^{\text{ét}}$ of $\mathbb{G}_{m,\overline{K}}$ are

$$U^{\text{ét}} = \mathbb{Q}_p(1) := \Big(\varprojlim_n \mu_{p^n}(\overline{K})\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

(c) Show that the Kummer map

$$\kappa\colon \mathbb{G}_m(K) \to \mathrm{H}^1(G_K, \mathbb{Q}_p(1))$$

can be identified with the natural map

$$K^\times \to \Big(\varprojlim_n K^\times / K^{\times p^n}\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For any prime ℓ consider the diagram

$$\begin{array}{ccc} \mathbb{G}_m(\mathbb{Z}) & \longrightarrow & \mathbb{G}_m(\mathbb{Z}_\ell) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_\ell} \\ \mathrm{H}^1(G_\mathbb{Q}, \mathbb{Q}_p(1)) & \xrightarrow{\mathrm{loc}_\ell} & \mathrm{H}^1(G_\ell, \mathbb{Q}_p(1)). \end{array}$$

(d) Describe the cohomology scheme $\mathrm{H}^1(G_\ell, \mathbb{Q}_p(1))$. Distinguish the cases $\ell \neq p$ and $\ell = p$.

(e) Define the Selmer scheme $\mathrm{Sel}_\infty(\mathbb{G}_m)$ as the scheme representing the subfunctor of $\mathrm{H}^1(G_\mathbb{Q}, \mathbb{Q}_p(1))$ consisting of classes α such that for all primes ℓ, the localisation $\mathrm{loc}_\ell(\alpha)$ is contained in $j_\ell(\mathbb{G}_m(\mathbb{Z}_\ell))^{\mathrm{Zar}}$. Moreover, let $\mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)) \coloneqq j_p(\mathbb{G}_m(\mathbb{Z}_p))^{\mathrm{Zar}}$. Show that the Chabauty–Kim diagram

$$\begin{array}{ccc} \mathbb{G}_m(\mathbb{Z}) & \longrightarrow & \mathbb{G}_m(\mathbb{Z}_p) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_p} \\ \mathrm{Sel}_\infty(\mathbb{G}_m) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)) \end{array}$$

can be identified with

$$\begin{array}{ccc} \mathbb{G}_m(\mathbb{Z}) & \longrightarrow & \mathbb{G}_m(\mathbb{Z}_p) \\ \downarrow & & \downarrow{\scriptstyle \log} \\ \{0\} & \xrightarrow{\mathrm{loc}_p} & \mathbb{Q}_p \end{array}$$

where the right vertical map is the p-adic logarithm $\log\colon \mathbb{Z}_p^\times \to \mathbb{Q}_p$.

(f) The Chabauty–Kim locus $\mathbb{G}_m(\mathbb{Z}_p)_\infty$ is defined as the set of $x \in \mathbb{G}_m(\mathbb{Z}_p)$ such that $j_p(x)$ lies in the scheme-theoretic image of $\mathrm{Sel}_\infty(\mathbb{G}_m)$ under the localisation map loc_p. This is a subset of $\mathbb{G}_m(\mathbb{Z}_p)$ containing $\mathbb{G}_m(\mathbb{Z})$ by construction. Determine $\mathbb{G}_m(\mathbb{Z}_p)_\infty$ depending on p. Show that

$$\mathbb{G}_m(\mathbb{Z}_p)_\infty = \{\pm 1\}$$

for a suitable choice of p, proving that $\mathbb{G}_m(\mathbb{Z}) = \{\pm 1\}$.