

LINEAR AND QUADRATIC CHABAUTY

FOR AFFINE HYPERBOLIC CURVES

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X/\mathbb{Q} smooth proj. curve of genus $g \geq 2$

p prime of good red., $U^{\text{ét}} \twoheadrightarrow U$ $G_{\mathbb{Q}}$ -equiv. quotient

Chabauty - Kim diagram

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) \\ \downarrow j & & \downarrow j_p \\ \text{Sel}_U(X) & \xrightarrow{\text{loc}_p} & H_f^1(G_p, U) \end{array}$$

Chabauty - Kim locus

$$X(\mathbb{Q}_p)_U := \{x \in X(\mathbb{Q}_p) : j_p(x) \in \text{loc}_p(\text{Sel}_U(X))\}$$

$$X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)_U \subseteq X(\mathbb{Q}_p)$$

Conj: $\# X(\mathbb{Q}_p)_U < \infty$ for U sufficiently large

• Linear Chabauty: $U = U_1^{\text{ét}} = (U^{\text{ét}})^{\text{ab}} \twoheadrightarrow X(\mathbb{Q}_p)_1$

Thm: Let $r_p := \text{rk}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(\text{Jac}_X)$ ($= r$ under Tate-Shafarevich conj.).

If $g - r_p > 0$, then $\# X(\mathbb{Q}_p)_1 < \infty$.

• Quadratic Chabauty: $U = U_{\text{qc}} = \text{certain intermediate quotient } U_2^{\text{ét}} \twoheadrightarrow U_{\text{qc}} \twoheadrightarrow U_1^{\text{ét}} \twoheadrightarrow X(\mathbb{Q}_p)_{\text{qc}}$

Thm: (Balakrishnan - Dogra)

Let $\beta_f := \text{rk } NS(\text{Jac}_X) + \text{rk } NS(\text{Jac}_{X_{\bar{\mathbb{Q}}}})^{\sigma=-1}$, $\sigma = \text{complex conj.}$

If $g + \beta_f - 1 - r_p > 0$, then $\# X(\mathbb{Q}_p)_{\text{qc}} < \infty$.

Aim: generalise this to affine hyperbolic curves

Setup: Y/\mathbb{Q} affine hyperbolic curve, $Y = X \setminus D$ with X projective, $n := \#D(\bar{\mathbb{Q}})$

- Ex:
- $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$,
 - $Y = E \setminus \{O\}$ with E elliptic curve,
 - $Y: y^2 = f(x)$ affine hyperell. curve

S finite set of primes

$\mathbb{Z}_S = \{x \in \mathbb{Q} : v_\ell(x) \geq 0 \ \forall \ell \notin S\}$ ring of S -integers

$Y = X \setminus D$ regular model of $Y = X \setminus D$ over \mathbb{Z}_S

Thm: (Siegel, Faltings)

$$\#Y(\mathbb{Z}_S) < \infty.$$

Variant of Chabauty-Kim for S -integral points:

$p \notin S$ s.t. $X_{\mathbb{F}_p}$ and $D_{\mathbb{F}_p}$ smooth, choose base point in $Y(\mathbb{Z}_S)$ for $U^{\text{ét}}$,

$U^{\text{ét}} \twoheadrightarrow U$ $G_{\mathbb{Q}}$ -equiv. quotient.

Chabauty-Kim diagram:

$$\begin{array}{ccc} Y(\mathbb{Z}_S) & \longrightarrow & Y(\mathbb{Z}_p) \\ \downarrow j & & \downarrow j_p \\ \text{Sel}_{S,u}(Y) & \xrightarrow{\text{loc}_p} & H_f^1(G_p, U) \end{array}$$

Chabauty-Kim locus:

$$Y(\mathbb{Z}_p)_{S,u} := \{y \in Y(\mathbb{Z}_p) : j_p(y) \in \text{loc}_p(\text{Sel}_{S,u}(Y))\}$$

- Linear Chabauty: $U = U_1^{\text{ét}} \rightsquigarrow Y(\mathbb{Z}_p)_{S,1}$
- Quadratic Chabauty: construct suitable intermediate quotient $U_2^{\text{ét}} \twoheadrightarrow U_{\text{QC}} \twoheadrightarrow U_1^{\text{ét}} \rightsquigarrow Y(\mathbb{Z}_p)_{S,\text{QC}}$

Invariants attached to (Y, S, ρ) :

- $r_p = \text{rk}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(\text{Jac}_X)$ p^∞ -Selmer rank
- $g = \text{genus of } X$
- $\beta_f = \text{rk NS}(\text{Jac}_X) + \text{rk NS}(\text{Jac}_{X_{\bar{\mathbb{Q}}}})^{\sigma = -1}$
- $n = \#D(\bar{\mathbb{Q}})$ number of geom. pts at infinity
 $= n_1 + 2n_2$ with $n_1 = \#D(\mathbb{R})$, $2n_2 = \#(D(\mathbb{C}) \setminus D(\mathbb{R}))$
- $b := \#|D| + n_2 - 1$ (≥ 0 since Y affine)
- $s := \#S$

Theorem A: (Leonhardt - L. - Müller)

- (1) If $\alpha_1 := g - r_p + b - s > 0$, then $\#Y(\mathbb{Z}_p)_{s,1} < \infty$.
- (2) If $\alpha_2 := \alpha_1 + \beta_f > 0$, then $\#Y(\mathbb{Z}_p)_{s,\text{qc}} < \infty$.

Theorem B: (LLM)

- (1) If $\beta_1 := \frac{1}{2}g(g+3) - \frac{1}{2}r(r+3) + b - s > 0$, then
- $$\#Y(\mathbb{Z}_p)_{s,1} \leq k_p \cdot \prod_{l \in S} (n_l + 1) \cdot \prod_{l \notin S} n_l \cdot \#Y(\mathbb{F}_p) \cdot (4g + 2n - 2)^2 (g+1).$$
- (2) If $\beta_2 := \beta_1 + \beta_f > 0$, same bound for $Y(\mathbb{Z}_p)_{s,\text{qc}}$.

Here: $k_p := 1 + \frac{p-1}{(p-2)\log(p)}$ (p odd), $k_2 := 2 + \frac{2}{\log(2)}$.

$n_l := \#$ inved. pts of the mod- l special fibre of $X_{\mathbb{F}_l}$ (if $l \notin S$)
 resp. of the minimal regular normal crossings model of (X, D)
 over \mathbb{Z}_l (if $l \in S$)

Ranks: • Have similar results for $Y(\mathbb{Z}_p)_{s, \text{ut} \geq -2}$, $U_2^{\text{ét}} \rightarrow U_{\text{ut} \geq -2} \rightarrow U_{\text{qc}} \rightarrow U_1^{\text{ét}}$.

$$h_{\text{BK}} := \dim_{\mathbb{Q}_p} H_f^1(G_{\mathbb{Q}}, \text{Hom}(\Lambda^2 V_p \text{Jac}_X, \mathbb{Q}_p(1)))$$

(= 0 conjecturally by Bloch-Kato)

C. If $g^2 - r_p + \beta + b - s > 0$, then $\#Y(\mathbb{Z}_p)_{S, ut \geq -2} < \infty$

D. If $\frac{1}{2}g(3g+1) - \frac{1}{2}r_p(r_p+3) + \beta + b - s - h_{Bk} > 0$,
 then $Y(\mathbb{Z}_p)_{S, ut \geq -2}$ satisfies bound from Thm B.

- "Balakrishnan-Dogra trick" \leadsto CK loci $Y(\mathbb{Z}_S)_{S,U}^{BD}$ where Theorems hold with r instead of r_p
- Belts-Cornin-Leonhardt: Bound on $Y(\mathbb{Z}_S)_\infty$ assuming Tate-Shafarevich + Bloch-Kato.

1. Refined Selmer schemes

For all l have diagram

$$\begin{array}{ccc} Y(\mathbb{Z}_S) & \longrightarrow & Y(\mathbb{Q}_l) \\ \downarrow j & & \downarrow j_l \\ H^1(G_{\mathbb{Q}}, U) & \xrightarrow{\text{loc}_l} & H^1(G_l, U) \end{array}$$

can replace $Y(\mathbb{Q}_l)$ with $Y(\mathbb{Z}_l)$ for $l \notin S$

Selmer functor: $R \mapsto \{ \alpha \in H^1(G_{\mathbb{Q}}, U) : \text{loc}_l(\alpha) \in \begin{cases} j_l(Y(\mathbb{Z}_l))^{2ar}, & l \notin S, \\ j_l(Y(\mathbb{Q}_l))^{2ar}, & l \in S. \end{cases} \}$

This is representable by the (refined) Selmer scheme $\text{Sel}_{S,U}(Y)$.
 \leadsto CK diagram

Say $x, y \in Y(\mathbb{Z}_S)$ have same reduction type if

$\forall l$: mod- l reductions on the same irred. cpt. or (if $l \in S$) the same cuspidal point.

reduction type: $\Sigma = (\Sigma_l)_{l \text{ prime}}$

Selmer scheme is union $\text{Sel}_{S,U}(Y) = \bigcup_{\Sigma} \text{Sel}_{\Sigma,U}(Y)$,

corresponding to $Y(\mathbb{Z}_S) = \bigsqcup_{\Sigma} Y(\mathbb{Z}_S)_{\Sigma} \leftarrow \text{points of red. type } \Sigma$

$$\rightarrow Y(\mathbb{Z}_p)_{S,U} = \bigcup_{\Sigma} Y(\mathbb{Z}_p)_{S,U,\Sigma}$$

Σ -refined CK diagram

$$\begin{array}{ccc} Y(\mathbb{Z}_S)_{\Sigma} & \hookrightarrow & Y(\mathbb{Z}_p) \\ \downarrow j & & \downarrow j_p \\ \text{Sel}_{\Sigma,U}(Y) & \xrightarrow{\text{loc}_p} & H_f^1(G_p, U) \end{array}$$

Strategy for finiteness of CK loci:

$$\text{if } \dim \text{Sel}_{\Sigma,U}(Y) < \dim H_f^1(G_p, U)$$

$\Rightarrow \text{loc}_p$ not dominant

$$\Rightarrow \exists 0 \neq f \text{ s.t. } f \circ \text{loc}_p = 0$$

$$\Rightarrow Y(\mathbb{Z}_p)_{S,U,\Sigma} \in V(f \circ j_p) \text{ finite}$$

Can compute dimensions using (abelian) Galois cohomology.

2. Weight filtrations on Selmer schemes

Betti's filtration $\dots \subseteq W_{-2}U \subseteq W_{-1}U = U$ by subgroup schemes

$$\text{s.t. } [W_{-i}U, W_{-j}U] \subseteq W_{-(i+j)}U \quad \forall i, j \geq 1.$$

$$\Rightarrow \text{gr}_{-k}^W U := W_{-k}U / W_{-k-1}U \text{ rep'n of } G_{\mathbb{Q}} \text{ on fin. dim. } \mathbb{Q}_p\text{-v.s.}$$

$$\text{Sel}_{\Sigma,U} \hookrightarrow \prod_{k=1}^{\infty} H_f^1(G_{\mathbb{Q}}, \text{gr}_{-k}^W U) \times \prod_{\ell \in S} \mathcal{O}_{\Sigma_{\ell}} \quad (\text{non-canonically})$$

\uparrow crystalline at p , unram. away from p \uparrow $\dim \leq 1$

$$\Rightarrow \dim \text{Sel}_{\Sigma,U} \leq s + \sum_{k=1}^{\infty} \dim_{\mathbb{Q}_p} H_f^1(G_{\mathbb{Q}}, \text{gr}_{-k}^W U),$$

$$H_f^1(G_p, U) \cong \prod_{k=1}^{\infty} H_f^1(G_p, \text{gr}_{-k}^W U) \quad (\text{non-canonically})$$

$$\Rightarrow \dim H_f^1(G_p, U) = \sum_{k=1}^{\infty} \dim_{\mathbb{Q}_p} H_f^1(G_p, \text{gr}_{-k}^W U)$$

• Linear Chabauty: $U = U_1^{\text{ét}} = U_Y^{\text{ab}}$

$Y \hookrightarrow X$ induces $U_Y^{\text{ab}} \rightarrow U_X^{\text{ab}} = V_p \text{Jac}_X := \left(\varprojlim_n \text{Jac}_X(\bar{\mathbb{Q}})[p^n] \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$$1 \rightarrow \underbrace{\mathbb{Q}_p(1)^{D(\bar{\mathbb{Q}})} / \mathbb{Q}_p(1)}_{\text{gr}_{-2}^w =: I} \rightarrow U_Y^{\text{ab}} \rightarrow \underbrace{V_p \text{Jac}_X}_{\text{gr}_{-1}^w} \rightarrow 1$$

Can compute local and global Galois coh:

$$\text{wt } -1: \dim H_f^1(G_{\mathbb{Q}}, V_p \text{Jac}_X) = r_p$$

$$\dim H_f^1(G_p, V_p \text{Jac}_X) = g$$

$$\text{wt } -2: \dim H_f^1(G_{\mathbb{Q}}, I) = n_1 + n_2 - \#\mathcal{D}$$

$$\dim H_f^1(G_p, I) = n - 1$$

If $\alpha_1 > 0$ in Thm A

$$\Rightarrow \dim \text{Sel}_{\Sigma, u} < \dim H_f^1(G_p, U)$$

$$\Rightarrow \#Y(\mathbb{Z}_p)_{\Sigma, 1} < \infty$$

• Quadratic Chabauty: Construction of U_{QC}

$$1 \rightarrow \wedge^2 V_p \text{Jac}_X \oplus I \rightarrow U_Y / W_{-3} U_Y \rightarrow V_p \text{Jac}_X \rightarrow 1$$

$$\begin{array}{ccc} \downarrow \text{max. Artin-Tate} \\ \text{quotient} & \downarrow & \parallel \\ 1 \rightarrow (\mathbb{Q}_p \otimes \text{NS}(\text{Jac}_{X_{\bar{\mathbb{Q}}}})^{\vee}(1)) \oplus I \rightarrow U_{\text{QC}} \rightarrow V_p \text{Jac}_X \rightarrow 1 \end{array}$$

$$1 \rightarrow (\mathbb{Q}_p \otimes \text{NS}(\text{Jac}_{X_{\bar{\mathbb{Q}}}})^{\vee}(1)) \oplus I \rightarrow U_{\text{QC}} \rightarrow V_p \text{Jac}_X \rightarrow 1$$

\uparrow

can compute its Galois coh.

3. Bounding Ck loci

Weight filtration on $\mathcal{O}(\text{Sel}_{\Sigma, u})$ and $\mathcal{O}(H_f^1(G_p, U))$ by fin. dim. subspaces, preserved by $\text{loc}_p^{\#}: \mathcal{O}(H_f^1) \rightarrow \mathcal{O}(\text{Sel}_{\Sigma, u})$

Assume $\dim W_m \mathcal{O}(\text{Sel}_{\Sigma, u}) < \dim W_m \mathcal{O}(H_f^1)$

$$\Rightarrow \exists 0 \neq f \in W_m \mathcal{O}(H_f^1) \text{ s.t. } \text{loc}_p^\# f = 0$$

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$\Rightarrow f \circ j_p: Y(\mathbb{Z}_p) \rightarrow \mathbb{Q}_p$ is "Coleman algebraic function of weight $\leq u$ "

Betts: bound on number of zeroes in each residue disc

\Rightarrow get bound on $Y(\mathbb{Z}_p)_{s,u,\Sigma}$ and $Y(\mathbb{Z}_p)_{s,u}$

In our theorems we get weight 2 functions

\rightarrow sum of double and single Coleman integrals and rat. function

Compute $\dim W_m \mathcal{O}(Z)$ via Hilbert series.

$$HS_Z(t) := \sum_{i=0}^{\infty} \dim \text{gr}_i^W \mathcal{O}(Z) t^i \in \mathbb{N}_0[[t]]$$

$$\text{Betts: } HS_{\text{rel}_{\Sigma,u}}(t) \lesssim (1-t^2)^{-s} \prod_{k=1}^{\infty} (1-t^k)^{-\dim H_f^1(G_{\mathbb{Q}}, \text{gr}_{-k}^W U)} =: HS_{\text{glob}}(t)$$

$$HS_{H_f^1}(t) = \prod_{k=1}^{\infty} (1-t^k)^{-\dim H_f^1(G_p, \text{gr}_{-k}^W U)} =: HS_{\text{loc}}(t)$$

For $U = U_1^{\text{ét}}$:

$$HS_{\text{glob}}(t) = 1 + r_p t + (s + n_1 + n_2 - \#(D) + \frac{1}{2} r_p (r_p + 1)) t^2 + \dots$$

$$HS_{\text{loc}}(t) = 1 + g t + (n - 1 + \frac{1}{2} g (g + 1)) t^2 + \dots$$

If $\beta_1 > 0$ in Thm B

$$\Rightarrow t^1\text{-coeff of } HS_{\text{glob}}(t) < t^2\text{-coeff of } HS_{\text{loc}}(t)$$

$\Rightarrow \exists$ nonzero Coleman alg. function $Y(\mathbb{Z}_p) \rightarrow \mathbb{Q}_p$ of weight ≤ 2
vanishing on $Y(\mathbb{Z}_p)_{s,1,\Sigma}$

\Rightarrow bound on $\# Y(\mathbb{Z}_p)_{s,1,\Sigma}$ and $\# Y(\mathbb{Z}_p)_{s,1}$