COMPUTATIONS IN MIXED ELLIPTIC MOTIVIC CHABAUTY–KIM: INITIAL REPORT

DAVID CORWIN, MARTIN LÜDTKE

1. Introduction

1.1. **Effective Siegel–Faltings.**

Let X be a smooth proper curve of genus $g \geq 2$ over a number field k. The theorem of Faltings states that $X(k)$ is finite. A major open question is to find an algorithm for determining the finite set $X(k)$ given X/k. More generally, the combination of the theorems of Faltings, Siegel, and Mahler imply that whenever X is a smooth curve with negative (geometric) Euler characteristic, and S is a finite set of places of k, we have $\mathcal{X}(\mathcal{O}_{k,S})$ finite, for any $\mathcal{O}_{k,S}$-model[i] $\mathcal{X}$ of X.

This divides into three major cases based on the genus g of the smooth projective model of X:

- $g = 0$: X is an open subscheme of $\mathbb{P}^1$ with at least three punctures
- $g = 1$: X is an affine open subscheme of an elliptic curve
- $g \geq 2$: X is a projective curve or an open subscheme thereof, and the set of rational points is finite

This focuses on $g = 1$.

1.2. **Non-Abelian Chabauty's Method.**

The non-abelian Chabauty's method of Minhyong Kim [Kim05, Kim09], also known as Chabauty–Kim, addresses the problem of determining $\mathcal{X}(\mathcal{O}_{k,S})$ effectively. For $\mathcal{X}$ as above, $\mathfrak{p}$ a maximal ideal of $\mathcal{O}_{k,S}$, an integer n,[ii] and a basepoint $b \in \mathcal{X}(\mathcal{O}_{k,S})$, Kim constructs a diagram:

$$\begin{tikzcd}
\mathcal{X}(\mathcal{O}_{k,S}) \arrow[r] \arrow[d, "\kappa"] & \mathcal{X}(\mathcal{O}_{\mathfrak{p}}) \arrow[d, "\kappa_{\mathfrak{p}}"'] \arrow[dr, "\int_{\mathfrak{p}}"] & \\
\mathrm{Sel}_{S,n}(\mathcal{X}) \arrow[r, "\mathrm{loc}_n"'] & \mathrm{Sel}_n(\mathcal{X}/\mathcal{O}_{\mathfrak{p}}) \arrow[r, "\sim", "\log_{\mathrm{BK}}"'] & U_n^{\mathrm{dR}}/F^0U_n^{\mathrm{dR}}
\end{tikzcd} \tag{1}$$

The set $\mathrm{Sel}_{S,n}(\mathcal{X})$ is a global non-abelian Galois cohomology set of the form $H^1_{f,S}(G_k; U_n)$ or a finite union thereof,[iii] where f refers to the Selmer conditions of Bloch–Kato, and U_n is the nth descending central series quotient of the $\mathbb{Q}_p$-unipotent geometric fundamental group of X (based at b). More details may be found in [Cor20, §4].

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[i] We define an $\mathcal{O}_{k,S}$-model as a finite type, separated, faithfully flat scheme $\mathcal{X}$ over $\mathcal{O}_{k,S}$ with an isomorphism $\mathcal{X}_k \to X$.

[ii] Often called the *depth*, although this conflicts with the notion of *depth* in the theory of multiple zeta values, so we prefer the term *level*.

[iii] Technically, one must either expand S to include all places of bad reduction of X as in [Kim09] or take a finite union of twists as in [BD20].

We define the *Chabauty–Kim locus*:

$$\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n} := \kappa_{\mathfrak{p}}^{-1}(\mathrm{Im}(\mathrm{loc}_n)) = \int_{\mathfrak{p}}^{-1}(\mathrm{Im}(\log_{\mathrm{BK}} \circ \mathrm{loc}_n)).$$

The *Chabauty–Kim ideal*

$$\mathcal{I}_{S,n} = \mathcal{I}_{S,n}(\mathcal{X})$$

of regular functions vanishing on the image of $\log_{\mathrm{BK}} \circ \mathrm{loc}_n$ pulls back to a set $\int_{\mathfrak{p}}^{\#}(\mathcal{I}_{S,n})$ of functions on $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})$ vanishing on $\mathcal{X}(\mathcal{O}_{k,S})$, with $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}$ defined as its set of common zeroes.

The main result of this paper is to compute $\mathcal{I}_{S,n}(\mathcal{X})$ and $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}$ in cases not previously accessible to computation.

1.3. Application to Effective Siegel–Faltings.

We recall some of the motivation for computing $\mathcal{I}_{S,n}(\mathcal{X})$ and $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}$; specifically, how it relates to the problem of Effective Siegel–Faltings introduced in §1.1.

When

$$\dim_{\mathbb{Q}_p} \mathrm{Sel}_{S,n}(\mathcal{X}) < \dim_{\mathbb{Q}_p} \mathrm{Sel}_n(\mathcal{X}/\mathcal{O}_{\mathfrak{p}}), \tag{2}$$

the set $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}$ is finite, a consequence of the fact that $\kappa_{\mathfrak{p}}$ has Zariski dense image ([Kim09, Theorem 1]).

Kim shows ([Kim09, Proposition 2]) that this inequality holds for sufficiently large n if a part of the Bloch–Kato Conjecture holds.

The following appears as [BDCKW18, Conjecture 3.1] for $S = \emptyset$ and in [BDCKW18, §8] as a remark about what one "might conjecture":

Conjecture 1.1 (Kim's Conjecture). *For $k = \mathbb{Q}$, a regular minimal model*[iv] *$\mathcal{X}$ and n sufficiently large, we have*

$$\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n} = \mathcal{X}(\mathcal{O}_{k,S}).$$

This conjecture implies that if we can compute $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}$ up to arbitrary p-adic precision for all n, then there is an effective version of Faltings's Theorem. More precisely, if we have a subset F of $\mathcal{X}(\mathcal{O}_{k,S})$, then to check that $F = \mathcal{X}(\mathcal{O}_{k,S})$, we need only find some n for which $|\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}| = |F|$. Given a collection of p-adic analytic functions, the theory of Newton polygons then allows us to determine the total number of common zeroes.[v] Thus effective Faltings over $\mathbb{Q}$ is reduced modulo Conjecture 1.1 to the problem of computing, up to sufficient p-adic precision, the set of functions on U_n/F^0U_n that vanish on the image of $\log_{\mathrm{BK}} \circ \mathrm{loc}_n$.

2. THE MAIN RESULT

Our computations of $\mathcal{I}_{S,n}(\mathcal{X})$ and $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}$ are in the case $n = 3$, and $\mathcal{X}$ is a punctured elliptic curve over $\mathcal{O}_{k,S}$ with $k = \mathbb{Q}$ and $|S| = 1$. While we studied multiple elliptic curves, primes p, and sets S, we introduce some notation common to them all.

[iv] Let g denote the genus of the smooth projective closure $\overline{X}$ of X. Then $\mathcal{X}$ is a *regular minimal model* if it is the complement of a reduced horizontal divisor in the regular minimal model $\overline{\mathcal{X}}$ of $\overline{X}$ over $\mathcal{O}_{k,S}$ (resp. in $\mathbb{P}^1/\mathcal{O}_{k,S}$) when $g \geq 1$ (resp. when $g = 0$).

[v] Technically it allows only to determine the number of zeroes *with multiplicity*, but a stronger version of Conjecture 1.1 resolves this by stating that $\mathcal{X}(\mathcal{O}_{\mathfrak{p}})_{S,n}$ as an analytic space is reduced for sufficiently large n.

Let $\mathcal{E}$ be the minimal Weierstrass model of an elliptic curve over $\mathbb{Z}[1/S]$ with identity section $O \in \mathcal{E}(\mathbb{Z}[1/S])$, let $\mathcal{X} = \mathcal{E}' := \mathcal{E} \setminus \{O\}$, let $E := \mathcal{E}_{\mathbb{Q}}$ and $X = E' = \mathcal{X}_{\mathbb{Q}}$. We define the following Coleman analytic functions on $\mathcal{X}(\mathbb{Z}_p)$ via iterated Coleman integrals.

We first set $\omega_0 := \mathrm{d}x/y$, $\omega_1 := x\,\mathrm{d}x/y$, both in $\Omega^1(X)$. We then define, for $z \in \mathcal{E}(\mathbb{Z}_p)$:

- $J_1(z) := \int_O^z \omega_0$
- $J_2(z) := \int_O^z \omega_0\omega_1$
- $J_3(z) := \int_O^z \omega_0\omega_1\omega_0$
- $J_4(z) := \int_O^z \omega_0\omega_1\omega_1 - 2\int_O^z \omega_1$

Assume that E has p-Selmer rank 1, let α denote a choice of component of the Néron model of E at each place of $\mathbb{Q}$ (following the method of [BD20], as explained in [Cor21, §3.5]), and let $S = \{\ell\}$ for some prime $\ell \neq p$. Then by [Cor21, Theorem 1.4], assuming $H^1_f(G_{\mathbb{Q}}; V_p(E)(-1)) = 0$, there is a nonzero function of the form

$$c_1 J_4 + c_2 J_3 + c_3 J_1 J_2 + c_4 J_1^3 + c_5 J_1 \tag{3}$$

vanishing on the subset $\mathcal{E}'(\mathbb{Z}[1/S])_\alpha$ of $\mathcal{E}'(\mathbb{Z}[1/S])$ reducing to α at each bad prime of $\mathcal{E}$, in which not all c_i are zero. As explained in [Cor21, §2.4], one may verify $H^1_f(G_{\mathbb{Q}}; V_p(E)(-1)) = 0$ by computing $L_p(E, 2)$ and checking that it is nonzero. Note that [Cor21, Theorem 1.4] depends crucially on the main result of [CDC26].

Our main result is to determine such a function and compute its zero set in a few cases.

Theorem 2.1. *For E the elliptic curve $y^2 = x^3 + x^2 - 9x + 7$ (Cremona label '128a2'), $S = \{2\}$, and $p = 3, 5, 7$, we have a non-zero function of the form (3) in $\mathcal{I}_{\{2\},3}(\mathcal{X})$, with coefficients $c_1, \ldots, c_5$ given as follows:*

	$p = 3$	$p = 5$	$p = 7$
c_1	1	1	1
c_2	$123659722/3^2 + O(3^{15})$	$202228537903/5^2 + O(5^{15})$	$24079823184153/7 + O(7^{15})$
c_3	$113658899/3^2 + O(3^{15})$	$106941634807/5^2 + O(5^{15})$	$27884130819516/7 + O(7^{15})$
c_4	$667280621/3^4 + O(3^{15})$	$2783496064817/5^3 + O(5^{15})$	$150212174658502/7^2 + O(7^{15})$
c_5	$65854279/3^2 + O(3^{15})$	$64558043327/5 + O(5^{15})$	$4066190634281 + O(7^{15})$

The Chabauty–Kim locus $\mathcal{X}(\mathbb{Z}_p)_{\{2\},3}$ consists of the 13 known points

$$(-3,-4), (-3,4), (-1,-4), (-1,4), (1,0), (2,-1), (2,1), (3,-4), (3,4),$$
$$(29/4, -155/8), (29/4, 155/8), (19,-84), (19,84),$$

along with the additional p-adic points with x-coordinates as follows:

$p = 3$	$p = 5$	$p = 7$
$5830650 + O(3^{15})$	$166703174 + O(5^{15})$	$-1 + 2\sqrt{2}$
	$1482191568 + O(5^{15})$	$-1 - 2\sqrt{2}$
	$9650625907 + O(5^{15})$	$574758317373 + O(7^{15})$
	$25074712578 + O(5^{15})$	$1162578601155 + O(7^{15})$
		$2351419738405 + O(7^{15})$
		$2597613053696 + O(7^{15})$
		$3582802419202 + O(7^{15})$
		$4286473007874 + O(7^{15})$

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D.C. DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY. Email: `corwin_david@yahoo.com`
M.L. DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OLDENBURG / BGU. Email: `martin.luedtke@uol.de`