Winter workshop Chabauty-Kim, Heidelberg, 14.-16.2.2024

Foundations of Chabauty-Kim

$X/\mathbb{Q}$ sm. proj. curve of genus $g \geq 2$

<u>Thm:</u> (Faltings 1983)

$\# X(\mathbb{Q}) < \infty$

<u>Q:</u> ("Effective Mordell")

- How to find $X(\mathbb{Q})$?
  Unknown whether a general algorithm exists.
- Bounds on $\# X(\mathbb{Q})$?
  $\hookrightarrow$ <u>Uniformity Conjecture:</u> (Caporaso-Harris-Mazur)
  $\exists\, N(g)$ s.t. $\# X(\mathbb{Q}) < N(g)$ for all X of genus g

Chabauty-Kim theory: p-adic approach adressing these, still largely conjectural

1. Method of Chabauty (1941) and Coleman (1985)

$J := \mathrm{Jac}_X$ — ab. var $/\mathbb{Q}$ of dim. g

$J(\mathbb{Q})$ is f.g. ab. gp. (Mordell-Weil)

$r := \mathrm{rk}_{\mathbb{Z}}\, J(\mathbb{Q}) \quad \rightsquigarrow \quad J(\mathbb{Q}) \cong \mathbb{Z}^r \oplus (\text{finite})$

Assume $X(\mathbb{Q}) \neq \emptyset$ and fix $b \in X(\mathbb{Q})$.

$\rightsquigarrow$ Abel-Jacobi map $AJ: X \longrightarrow J, \quad P \longmapsto [P]-[b]$

Choose auxiliary prime p of good reduction for X.

Have commutative diagram

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r, hook] \arrow[d, "AJ"'] & X(\mathbb{Q}_p) \arrow[d, "AJ_p"] \\ J(\mathbb{Q}) \arrow[r, hook] & J(\mathbb{Q}_p) \end{tikzcd}$$

$$\Rightarrow X(\mathbb{Q}) \subseteq AJ_p^{-1}(J(\mathbb{Q})) \subseteq X(\mathbb{Q}_p)$$

Idea: find functions $0 \neq f: J(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ which vanish on $J(\mathbb{Q})$, then $f \circ AJ_b : X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ vanishes on $X(\mathbb{Q})$

This works if $\boxed{r < g}$. $\quad (*)$

Coleman integration: Fix $\omega \in H^0(X_{\mathbb{Q}_p}, \Omega^1) \cong H^0(J_{\mathbb{Q}_p}, \Omega^1) \ni \omega_J$.

ω_J translation-inv. on $J_{\mathbb{Q}_p}$ $\Rightarrow$ $\exists!$ anti-derivative $F_\omega : J(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ s.t.

- F_ω is a homom.
- near $0 \in J(\mathbb{Q}_p)$, F_ω is given by a convergent power series with $dF_\omega = \omega_J$

Write $\int_0^D \omega_J := F_\omega(D)$, $\quad \int_P^Q \omega := \int_0^{[Q]-[P]} \omega_J$ for $P, Q \in X(\mathbb{Q}_p)$.

Get homom. $\log : J(\mathbb{Q}_p) \longrightarrow H^0(J_{\mathbb{Q}_p}, \Omega^1)^\vee$, $\quad D \longmapsto \left[\omega \longmapsto \int_0^D \omega_J\right]$.

If $r < g$ $\Rightarrow$ $\log \overline{J(\mathbb{Q})} \subseteq H^0(J_{\mathbb{Q}_p}, \Omega^1)^\vee$ $\mathbb{Z}_p$-submodule of rank $r < g$

$\Rightarrow \exists\, 0 \neq \omega \in H^0(J_{\mathbb{Q}_p}, \Omega^1)$ s.t. F_ω vanishes on $J(\mathbb{Q})$

$\Rightarrow F_\omega \circ AJ_b : X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$, $\quad P \longmapsto \int_b^P \omega$ vanishes on $X(\mathbb{Q})$

$P \longmapsto \int_b^P \omega$ locally analytic (given by convergent power series on residue discs)

$\Rightarrow$ only finitely many zeroes

$\Rightarrow$ $X(\mathbb{Q})$ finite

Also get explicit bounds, e.g. $\# X(\mathbb{Q}) \leq \# X(\mathbb{F}_p) + (2g-2)$ if $p > 2g$.

Problem: method fails if $r \nless g$

Kim (2005): developed non-abelian generalisation to overcome this.

Central object: $\mathbb{Q}_p$-prounipotent étale fundamental group $U^{\text{ét}}$ of X

Chabauty–Kim diagram:

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, hook] \arrow[d, "j"] & X(\mathbb{Q}_p) \arrow[d, "j_p"] \arrow[dr, "j^{dR}"] & \\
\mathrm{Sel}_\infty(X) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U^{\text{ét}}) \arrow[r, "\log_{BK}", "\sim"'] & F^0\backslash U^{dR}
\end{tikzcd} \qquad \text{(CK)}$$

Properties:

- bottom row: affine $\mathbb{Q}_p$-schemes, maps are algebraic
- image of each residue disc $\mathrm{red}^{-1}(y) \subseteq X(\mathbb{Q}_p)$, $y \in X(\mathbb{F}_p)$, under j^{dR} is Zariski-dense
- Conjecturally, loc_p is non-dominant ($\Leftarrow$ Bloch-Kato conj.)

Method works as follows:

If loc_p non-dominant $\Rightarrow$ find $0 \neq f \in \mathcal{O}(F^0\backslash U^{dR})$ vanishing on image of $\mathrm{Sel}_\infty(X)$

$\Rightarrow f \circ j^{dR} : X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ non-trivial + analytic on residue discs (given by iterated Coleman integrals)

$\Rightarrow V(f \circ j^{dR}) \subseteq X(\mathbb{Q}_p)$ finite set containing $X(\mathbb{Q})$

Def: Chabauty-Kim locus

$$X(\mathbb{Q}_p)_\infty := j_p^{-1}(\mathrm{loc}_p(\mathrm{Sel}_\infty(X))) = \bigcap_{f \text{ as above}} V(f \circ j^{dR}) \subseteq X(\mathbb{Q}_p).$$

Note: $X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)_\infty$

Conj.: $X(\mathbb{Q}_p)_\infty$ is finite ($\Leftarrow$ Bloch-Kato)

Conj: (Kim's conjecture)

$$X(\mathbb{Q})_\infty = X(\mathbb{Q}).$$

Rmks:
- replace $U^{\acute{e}t}$ with n-step nilpotent quotient $U_n^{\acute{e}t}$ $\rightsquigarrow X(\mathbb{Q}_p)_n$
- roughly: "$n=1$" $\to$ Chabauty-Coleman
  "$n=2$" $\to$ quadratic Chabauty
- variant for S-integral points on affine hyperbolic curves

2. Prounipotent étale fundamental group

$X/\mathbb{Q}$, $b \in X(\mathbb{Q})$ as above

$\overline{\mathbb{Q}}/\mathbb{Q}$ alg. closure, $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$\rightsquigarrow$ $\pi_1^{\text{ét}}(X_{\bar{\mathbb{Q}}}, b)$ — "étale fundamental group"

It is a profinite group with cts. $G_{\mathbb{Q}}$-action.

$\mathrm{Cov}(X_{\bar{\mathbb{Q}}})$ — category of finite étale covers $Y \longrightarrow X_{\bar{\mathbb{Q}}}$

$F_b : \mathrm{Cov}(X_{\bar{\mathbb{Q}}}) \longrightarrow \mathrm{FinSet}$, $(Y \xrightarrow{f} X_{\bar{\mathbb{Q}}}) \mapsto f^{-1}(b)$ fibre functor

$\pi_1^{\text{ét}}(X_{\bar{\mathbb{Q}}}, b) := \mathrm{Aut}(F_b)$, acts compatibly on $f^{-1}(b)$ for all $(Y \xrightarrow{f} X_{\bar{\mathbb{Q}}}) \in \mathrm{Cov}(X_{\bar{\mathbb{Q}}})$

Comparison with topological π_1: (via Riemann's existence theorem)

$$\pi_1^{\text{ét}}(X_{\bar{\mathbb{Q}}}, b) = \text{profinite completion of } \pi_1^{\text{top}}(X(\mathbb{C}), b)$$
$$\cong \langle a_1, b_1, \dots, a_g, b_g \mid \prod_i [a_i, b_i] = 1 \rangle^{\wedge}$$

<u>Malčev completion:</u> Π profinite gp. p prime

The $\mathbb{Q}_p$-prounipotent completion of Π is the universal prounipotent algebraic gp $\Pi^{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ with a continuous homom.

$$\Pi \longrightarrow \Pi^{\mathbb{Q}_p}(\mathbb{Q}_p).$$

Think: $\Pi^{\mathbb{Q}_p} =$ "$\mathbb{Q}_p \underset{\hat{\mathbb{Z}}}{\otimes} \Pi$".

<u>Def:</u> $\pi_1^{\text{ét},\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}}, b) := \pi_1^{\text{ét}}(X_{\bar{\mathbb{Q}}}, b)^{\mathbb{Q}_p}$

"$\mathbb{Q}_p$-prounipotent étale fundamental group"

<u>Alternative definition via Tannakian formalism</u>

$\mathrm{Loc}^{\mathrm{un}}_{\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}}) :=$ category of unipotent $\mathbb{Q}_p$-local systems on $X_{\bar{\mathbb{Q}}}$,

i.e. étale-locally constant sheaves of $\mathbb{Q}_p$-v.s. on $X_{\bar{\mathbb{Q}},\text{ét}}$ which admit a filtration $0 \leq \mathrm{Fil}_0 E \leq \dots \leq \mathrm{Fil}_m E = E$ s.t.

$\mathrm{Fil}_i E / \mathrm{Fil}_{i-1} E \cong$ direct sum of copies of $\underline{\mathbb{Q}_p}$.

$\mathrm{Fib}_b : \mathrm{Loc}^{\mathrm{un}}_{\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}}) \longrightarrow \mathbb{Q}_p\text{-Vect}$, $E \mapsto E(b)$ fibre functor, preserves $\otimes$

$\pi_1^{\text{ét},\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}}, b) := \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Fib}_b)$ tensor-preserving automorphisms

<u>Notation:</u> $U^{\text{ét}} := \pi_1^{\text{ét},\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}}, b)$, $U_n^{\text{ét}} :=$ n-step nilpotent quotient

$G_{\mathbb{Q}}$ acts on $U^{\text{ét}}_{(n)}$. Fix $G_{\mathbb{Q}}$-equivariant quotient $U^{\text{ét}} \twoheadrightarrow U'$, e.g. $U' = U_n^{\text{ét}}$.

Path torsors: For $x \in X(\mathbb{Q})$ have

$$P^{ét}(x) := \pi_1^{ét,\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}};\, b, x) := \underline{\mathrm{Isom}}^{\otimes}(\mathrm{Fib}_b, \mathrm{Fib}_x)$$ "path torsor"

It is a $G_{\mathbb{Q}}$-equivariant $U^{ét}$-torsor over $\mathbb{Q}_p$.

Form pushout along $U^{ét} \twoheadrightarrow U'$

$$\rightsquigarrow P'(x) := P^{ét}(x)/\ker(U^{ét} \twoheadrightarrow U'),$$

$G_{\mathbb{Q}}$-equiv. U'-torsor

Can do the same locally for $\mathbb{Q}_\ell$-points (ℓ any prime):

Fix $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ with embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$

$\rightsquigarrow$ local Galois group $G_\ell := \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \leq G_{\mathbb{Q}}$

Have $\pi_1^{ét,\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}_\ell}, b) = \pi_1^{ét,\mathbb{Q}_p}(X_{\bar{\mathbb{Q}}}, b)$.

Given $x \in X(\mathbb{Q}_\ell)$ $\rightsquigarrow$ G_ℓ-equivariant U'-torsor $P'(x)$

3. Spaces of torsors

G profinite gp, $U/\mathbb{Q}_p$ pro-unipotent group, $G \curvearrowright U$

<u>Def</u>: R $\mathbb{Q}_p$-algebra. A <u>G-equivariant U-torsor</u> P over R is an R-scheme P with a G-action and a G-equivariant right U-action $P \times U \longrightarrow U$ such that $P \to \mathrm{Spec}(R)$ is faithfully flat and the map

$$P \times U \longrightarrow P \times_R P, \qquad (\gamma, u) \mapsto (\gamma, \gamma u)$$

is an isomorphism.

<u>Rmk</u>: U prounipotent implies that all U-torsors are trivial as mere torsors, but not necessarily G-equivariantly.

Non-abelian group cohomology:

$c : G \to U(R)$ is a cocycle if $c(\sigma\tau) = c(\sigma) \cdot {}^{\sigma}c(\tau) \quad \forall \sigma, \tau \in G.$

c and c' are cohomologous ($c \sim c'$) if $\exists u \in U(R)$ s.t. $c'(\sigma) = u^{-1} c(\sigma)\, {}^{\sigma}u \quad \forall \sigma \in G$

$H^1(G, U(R)) := \{\text{cocycles } G \to U(R)\}/\sim$

Cohomology functor on $\mathbb{Q}_p$-algebras:

$$H^1(G, U) : R \mapsto H^1(G, U(R))$$

Prop: $\left(\begin{array}{c} G\text{-equivariant } U\text{-torsors} \\ \text{over } R \end{array} \right) / \text{iso} \quad \cong \quad H^1(G, U(R))$

Pf: P G-equiv. U-torsor

U pro-unipotent $\Rightarrow P(R) \neq \emptyset$

Choose $p_0 \in P(R)$. For $\sigma \in G$ $\exists!$ $c(\sigma) \in U(R)$ s.t. ${}^{\sigma}p_0 = p_0 \cdot c(\sigma)$.

Check: • $c : G \to U(R)$ is cocycle

• different choice of p_0 gives cohomologous cocycle

Conversely, given $c : G \to U(R)$ define torsor $P_c := U$ with c-twisted G-action $G \times U \to U$, $(\sigma, u) \mapsto {}^{\sigma}u \cdot c(\sigma)$. $\square$

Rmk: $1 \to A \to U \to U' \to 1$ central extension induces long exact sequence of pointed sets ending at $H^2(G, A)$

Thm: Assume U admits a separated G-stable filtration

$$U = W_{-1}U \trianglerighteq W_{-2}U \trianglerighteq \dots$$

with $[W_{-i}U, W_{-j}U] \subseteq W_{-i-j}U \quad \forall i, j \geq 1$. Write $V_n := gr^W_{-n}U$.

Note V_n is a $\mathbb{Q}_p$-v.s. Assume:

(1) $H^0(G, V_n) = 0$

(2) $H^1(G, V_n)$ is fin. dim.

for all $n \geq 1$. Then $H^1(G, U)$ is representable by an affine

$\mathbb{Q}_p$-scheme which is non-canonically isomorphic to a closed subscheme of $\prod_{n\geq 1} H^1(G, V_n)$.

<u>Proof of representability:</u> (Sketch)

Let $U_n := U/W_{-(n+1)}U$, then $U = \varprojlim_n U_n$.

$H^1(G,U) \xrightarrow{\sim} \varprojlim_n H^1(G,U_n)$ iso $\Rightarrow$ reduced to $H^1(G,U_n)$

Use induction on n.

Case $n=1$: $U_1 = V_1$ vector group, $U_1(R) = V_1 \otimes R$.

Show $H^1(G,V_1)\otimes R \longrightarrow H^1(G, V_1\otimes R)$ iso.

$\Rightarrow$ $H^1(G,V_1)$ represented by vector scheme $H^1(G,V_1)$

Inductive step $n \to n+1$.

Central extension

$$1 \longrightarrow V_n \longrightarrow U_n \longrightarrow U_{n-1} \longrightarrow 1$$

$\Rightarrow$ LES of functors on $\mathbb{Q}_p$-algebras

$$\begin{array}{l} 1 \longrightarrow H^0(G,V_n) \longrightarrow H^0(G,U_n) \longrightarrow H^0(G,U_{n-1}) \\ \longrightarrow H^1(G,V_n) \longrightarrow H^1(G,U_n) \longrightarrow H^1(G,U_{n-1}) \xrightarrow{\delta^1} \\ \longrightarrow H^2(G,V_n) \end{array}$$

- $H^1(G,U_{n-1})$ and $H^1(G,V_n)$ repr.

  $\Rightarrow$ $K := \ker(\delta^1)$ repr. by affine $\mathbb{Q}_p$-scheme

- $H^1(G,U_n) \longrightarrow K$ is surjective map of functors

  $K = \mathrm{Spec}(S)$, then $H^1(G,U_n)(S) \twoheadrightarrow K(S) = \mathrm{Hom}(S,S) \ni \mathrm{id}_S$ surj.

  $\rightsquigarrow$ get splitting $s: K \longrightarrow H^1(G,U_n)$

- $H^1(G,U_n) \to K$ is $H^1(G,V_n)$-torsor

  $\Rightarrow$ $K \times H^1(G,V_n) \xrightarrow{\sim} H^1(G,U_n)$ iso

  $\Rightarrow$ $H^1(G,U_n)$ representable $\square$

<u>Cor:</u> $X/\mathbb{Q}$, $U^{\text{ét}} \twoheadrightarrow U'$ as before, ℓ prime

$$\Rightarrow \quad R \mapsto \left(\begin{matrix} G_\ell\text{-equiv. } U'\text{-torsors} \\ \text{over } R \end{matrix} \right) \Big/ \text{iso} \;\cong\; H^1(G_\ell, U'(R))$$

is representable by affine $\mathbb{Q}_p$-scheme.

Denote it by $H^1(G_\ell, U')$.

<u>Pf:</u> (for $\ell \neq p$)

Check conditions of Theorem.

Define $W_\bullet U'$ via descending central series:

$$W_{-1} U' := U', \quad W_{-i+1} U' := [W_{-i} U', U'].$$

$V_n := \mathrm{gr}^W_n U'$. Claim: V_n is pure of weight $-n$, i.e., the eigenvalues α of any geom. Frobenius $F \in G_\ell$ are alg. $/\mathbb{Q}$ and $|\iota(\alpha)| = \ell^{-n/2}$ for all embeddings $\iota: \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$. Follows from:

- $V_1 = (U')^{ab} \twoheadleftarrow U^{\text{ét}, ab} = H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell}, \mathbb{Q}_p)^*$, pure of wt -1 by Riemann hypothesis for curves over finite fields
- $V_1^{\otimes n} \twoheadrightarrow V_n$, $v_1 \otimes \ldots \otimes v_n \longmapsto [v_1, [v_2, [\ldots, v_n]]\ldots]$

  $\Rightarrow V_n$ pure of wt $-n$

Claim $\Rightarrow H^0(G_\ell, V_n) = 0 \quad \forall n$

G_ℓ has "property (F)", i.e. $\exists$ only fin. many open subgroups of any given index

$\Rightarrow H^1(G_\ell, V_n)$ fin. dim. $\forall i$

$\Rightarrow H^1(G_\ell, U')$ representable by Thm $\square$

Get diagram for every prime ℓ

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r, hook] \arrow[d, "j"] & X(\mathbb{Q}_\ell) \arrow[d, "j_\ell"] \\ H^1(G_{\mathbb{Q}}, U') \arrow[r, "\mathrm{loc}_\ell"] & H^1(G_\ell, U') \end{tikzcd}$$

j, j_ℓ are "non-abelian Kummer maps" or "higher Albanese maps"

Selmer functor:

$$\mathrm{Sel}_{U'}(X): \quad R \mapsto \{\alpha \in H^1(G_{\mathbb{Q}}, U')(R) \mid \forall \ell: \ \mathrm{loc}_\ell(\alpha) \in j_\ell(X(\mathbb{Q}_\ell))^{\mathrm{Zar}}\}$$

Rmk:

- if $\ell \neq p$: $j_\ell(X(\mathbb{Q}_\ell))$ finite,
  if ℓ of good reduction: $j_\ell(X(\mathbb{Q}_\ell)) = \{*\}$
- if $\ell = p$: $j_p(X(\mathbb{Q}_p))^{\mathrm{Zar}} = H^1_f(G_p, U')$, $R \mapsto \ker\big(H^1(G_p, U'(R)) \to H^1(G_p, U'(R \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}))\big)$
  subscheme of "crystalline classes" — Fontaine's period ring

Thm: The Selmer functor is representable by an affine $\mathbb{Q}_p$-scheme.

This is the Selmer scheme $\mathrm{Sel}_{U'}(X)$.

(Pf sketch: for T finite set of primes,

$G_T :=$ Galois gp of max. ext'n of $\mathbb{Q}$ which is unram. outside T,

G_T (unlike $G_{\mathbb{Q}}$) has property (F),

$\mathrm{Sel}_{U'}(X) \subseteq H^1(G_T, U')$ for some T)

4. The de Rham fundamental group

$X/\mathbb{Q}_p$ for now

$MIC^{un}(X)$: Tannaka category of unipotent vector bundles w/ integrable connection.

objects: $(\mathcal{E}, \nabla)$ with $\mathcal{E}$ vector bundle on X,

$\nabla: \mathcal{E} \to \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ connection ($\nabla^2 = 0$ automatically)

$\exists$ filtration $\mathcal{E} = \mathcal{E}_n \supseteq \ldots \supseteq \mathcal{E}_0 = 0$ s.t. $\mathcal{E}_{i+1}/\mathcal{E}_i \cong (\mathcal{O}_X, d)$ unit object

$b \in X(\mathbb{Q}_p)$ defines fibre functor $\mathrm{Fib}_b: MIC^{un}(X) \to \mathbb{Q}_p\text{-}\mathrm{Mod}_{f.d.}$

$U^{dR} := \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Fib}_b)$ de Rham fundamental group

$P^{dR}(x,y) := \underline{\mathrm{Isom}}^{\otimes}(\mathrm{Fib}_x, \mathrm{Fib}_y)$ de Rham path space $(x, y \in X(\mathbb{Q}_p))$

<u>Iterated integrals:</u> Let $\omega_1, \dots, \omega_n \in H^0(X, \Omega^1)$, $\gamma \in P^{dR}(x,y)(R)$, R $\mathbb{Q}_p$-algebra.

$\rightsquigarrow$ define $\int_\gamma \omega_1 \cdots \omega_n \in R$ as follows

$$\mathcal{E}_{\underline{\omega}} := (\mathcal{O}_X^{n+1}, \nabla) \quad \text{with} \quad \nabla \begin{pmatrix} f_1 \\ \vdots \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_{n+1} \end{pmatrix} - \begin{pmatrix} 0 & \omega_1 & & & \\ & 0 & \omega_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \omega_n \\ & & & & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ \\ f_{n+1} \end{pmatrix}$$

$$\gamma_{\mathcal{E}_{\underline{\omega}}} : R^{n+1} = \mathrm{Fib}_x(\mathcal{E}_{\underline{\omega}}) \longrightarrow \mathrm{Fib}_y(\mathcal{E}_{\underline{\omega}}) = R^{n+1}$$

$$\int_\gamma \omega_1 \cdots \omega_n := e_1^\vee(\gamma_{\mathcal{E}_{\underline{\omega}}}(e_{n+1})) \in R$$

<u>Fact:</u> Every alg. function on $P^{dR}(x,y)$ is a $\mathbb{Q}_p$-linear combination of iterated integrals $\gamma \mapsto \int_\gamma \omega_1 \cdots \omega_n$.

<u>Iterated Coleman integrals:</u>

$X_0 :=$ special fibre of unique smooth model of X over $\mathbb{Z}_p$

$\rightsquigarrow$ Tannaka category $\mathrm{Isoc}^{un}(X_0)$ of unipotent isocrystals on X_0

$\rightsquigarrow$ crystalline fundamental gp U^{cris}, path space $P^{cris}(x,y)$

Comparison thm: $MIC^{un}(X) \simeq \mathrm{Isoc}^{un}(X_0)$

$\Rightarrow$ $U^{dR} \cong U^{cris}$, $P^{dR}(x,y) \cong P^{cris}(\bar{x}, \bar{y})$

$\rightsquigarrow$ Frob $\circlearrowright$ U^{dR}, $P^{dR}(x,y)$

<u>Thm:</u> (Besser)

For all $x, y \in X(\mathbb{Q}_p)$, $\exists!$ Frobenius-invariant path $p^{cr} \in P^{dR}(x,y)(\mathbb{Q}_p)$.

<u>Def:</u> Let $x, y \in X(\mathbb{Q}_p)$, $\omega_1, \dots, \omega_n \in H^0(X, \Omega^1)$.

$$\int_x^y \omega_1 \cdots \omega_n := \int_{p^{cr}} \omega_1 \cdots \omega_n \in \mathbb{Q}_p \qquad \underline{\text{iterated Coleman integral}}$$

<u>Rmk:</u> For x, y in the same residue disc, t unif. at x, can compute $\int_x^y \omega_1 \cdots \omega_n$ by expanding $\omega_i = f_i(t)\,dt$ as convergent power series and integrating iteratively: $\int_x^y \omega_1 \cdots \omega_n = \cdots \int f_{n-1}(t) \int f_n(t)\,dt \Big|_{t = t(y)}$

de Rham Kummer map

U^{dR} carries Hodge filtration $F_\bullet U^{dR}$ by subschemes and Frobenius autom.
$F_0 U^{dR} \leq U^{dR}$ subgp. $\rightsquigarrow$ $F^0\backslash U^{dR}$ right coset scheme

Define "admissible U^{dR}-torsors": carry compatible Hodge filtr. + Frob.

<u>Thm</u>: $(F^0\backslash U^{dR})(R) \cong \left(\begin{array}{c}\text{admissible } U^{dR}\text{-torsors} \\ \text{over } R\end{array}\right)/\text{iso}$

<u>Sketch</u>: P adm. U^{dR}-torsor / R

$\rightarrow$ $\exists\, p^H \in F_0 P(R)$, $\exists$ unique $p^{cr} \in P(R)^{Frob}$

$P \mapsto (p^H)^{-1} \cdot p^{cr} \in (F^0\backslash U^{dR})(R)$

$\square$

de Rham Kummer map:

$$j^{dR}: X(\mathbb{Q}_p) \longrightarrow F^0\backslash U^{dR}. \qquad x \longmapsto [P^{dR}(b,x)] = \underset{\substack{\uparrow \\ \text{Hodge path}}}{(p^H)^{-1}} \cdot \underset{\substack{\uparrow \\ \text{Frob.-inv. path}}}{p^{cr}}$$

pulling back $(\int \omega_1 \cdots \omega_n) \in \mathcal{O}(F^0\backslash U^{dR})$ along j^{dR} gives iterated Coleman integral

$$X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p. \qquad x \longmapsto \int_b^x \omega_1 \cdots \omega_n.$$