Refined Chabauty–Kim computations for the thrice-punctured line over $\mathbb{Z}[1/6]$

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Effective Mordell

$X/\mathbb{Q}$ smooth projective curve of genus $g \geq 2$

Mordell Conjecture (1922), Faltings's Theorem (1983)

$\#X(\mathbb{Q}) < \infty$

- Chabauty (1941): proved finiteness if $r := \operatorname{rk} \operatorname{Jac}_X(\mathbb{Q}) < g$
- Faltings (1983): proved Mordell in general

Open problem: How to determine $X(\mathbb{Q})$ in practice?

Can use computer search to list points in $X(\mathbb{Q})$ but how do we know we found them all?

Chabauty's proof can be made effective but the condition $r < g$ is not always satisfied

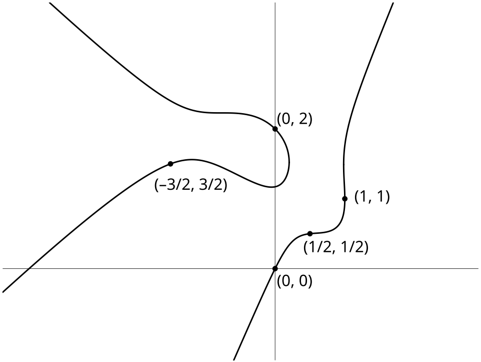
Example: the cursed curve

$$y^4 + 5x^4 - 6x^2y^2 + 6x^3 + 26x^2y + 10xy^2 - 10y^3 - 32x^2 - 40xy + 24y^2 + 32x - 16y = 0$$

It has rational points

$$(0,0), \quad (0,2), \quad (1,1), \quad (\tfrac{1}{2},\tfrac{1}{2}), \quad (-\tfrac{3}{2},\tfrac{3}{2})$$

but how do we know there are no others?
This question comes up in a question of Serre from 1972 about residual Galois representations of elliptic curves.



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Idea: develop "non-abelian" generalisation of Chabauty

- Kim (2005): Chabauty–Kim method (aka non-abelian Chabauty)

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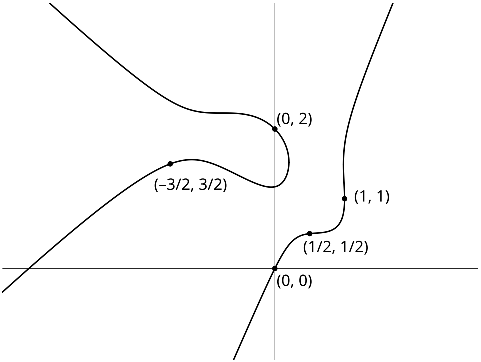

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The Chabauty–Kim method

For p a prime of good reduction, try to locate $X(\mathbb{Q})$ inside $X(\mathbb{Q}_p)$. Kim constructs a descending sequence of subsets

$$X(\mathbb{Q}_p) \supseteq X(\mathbb{Q}_p)_1 \supseteq X(\mathbb{Q}_p)_2 \supseteq \dots$$

all containing $X(\mathbb{Q})$. The set $X(\mathbb{Q}_p)_n$ is called the Chabauty–Kim locus of depth n.

- $X(\mathbb{Q}_p)_n$ is cut out inside $X(\mathbb{Q}_p)$ by p-adic analytic functions (more precisely: iterated Coleman integrals)
- $X(\mathbb{Q}_p)_1$ is finite if $r < g$ (Chabauty)
- $X(\mathbb{Q}_p)_2$ is finite if $r < g + \rho - 1$, where $\rho := \operatorname{rk} \mathrm{NS}(\mathrm{Jac}_X)$ (Quadratic Chabauty)
- Bloch–Kato *or* Fontaine–Mazur conjecture $\Rightarrow \#X(\mathbb{Q}_p)_n < \infty$ for $n \gg 0$

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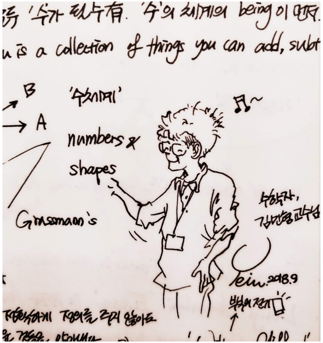
Kim's Conjecture

$X(\mathbb{Q}_p)_n = X(\mathbb{Q})$ for $n \gg 0$.

- Practical relevance: if true, can try to compute $X(\mathbb{Q})$ by computing $X(\mathbb{Q}_p)_n$ for $n = 1, 2, \ldots$
- Theoretical relevance: Kim's Conjecture implies local-to-global principle for Grothendieck's Section Conjecture (Betts–Kumpitsch–L.)



Computing $X(\mathbb{Q}_p)_n$ is hard!

Construction of Chabauty–Kim loci

Idea: Let U be a quotient of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of X and construct the Chabauty–Kim diagram via moduli spaces of U-torsors with Galois action

$$\begin{array}{ccc} X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) \\ \downarrow j & & \downarrow j_p \\ \mathrm{Sel}_U(X)(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, U)(\mathbb{Q}_p) \end{array}$$

Fact: loc_p is an algebraic map of affine $\mathbb{Q}_p$-schemes

Strategy:

- show that loc_p is not dominant (e.g., for dimension reasons)
- find $0 \neq f\colon \mathrm{H}^1_f(G_p, U) \to \mathbb{A}^1$ vanishing on $\mathrm{im}(\mathrm{loc}_p)$
- the pullback $f \circ j_p\colon X(\mathbb{Q}_p) \to \mathbb{Q}_p$ is a nonzero p-adic analytic function whose vanishing set is finite and contains $X(\mathbb{Q})$

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The thrice-punctured line

Today: compute some (refined) Chabauty–Kim loci in the best-understood example

$$\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}.$$

Setting:

- S: finite set of primes
- $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{\ell} : \ell \in S]$: ring of S-integers
- $X = \mathbb{P}^1_{\mathbb{Z}_S} \smallsetminus \{0, 1, \infty\}$: thrice-punctured line

We are interested in the S-integral points $X(\mathbb{Z}_S)$. S-unit equation:

$$x + y = 1 \quad \text{with } x, y \in \mathbb{Z}_S^\times$$

Solutions are $x \in \mathbb{Q}$ s.t. x and $1 - x$ are of the form $\pm \prod_{\ell \in S} \ell^{e_\ell}$ with $e_\ell \in \mathbb{Z}$.

Theorem (Siegel–Mahler, 1933)

$X(\mathbb{Z}_S)$ is finite.

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Some small sets S

- Example $S = \emptyset$:

$$X(\mathbb{Z}) = \emptyset$$

- Example $S = \{2\}$:

$$X(\mathbb{Z}[1/2]) = \left\{2, -1, \frac{1}{2}\right\}$$

- Example $S = \{2, 3\}$:

$$X(\mathbb{Z}[1/6]) = \left\{2, \frac{1}{2}, -1, 3, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, -\frac{1}{2}, -2, 4, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, -\frac{1}{3}, -3, 9, \frac{1}{9}, \frac{8}{9}, \frac{9}{8}, -\frac{1}{8}, -8\right\}$$

(Levi ben Gershon 1342, *The Harmony of Numbers*)

Chabauty–Kim for the thrice-punctured line

Let $p \notin S$, so that $X(\mathbb{Z}_S) \subseteq X(\mathbb{Z}_p)$. Have Chabauty–Kim loci

$$X(\mathbb{Z}_p) \supseteq X(\mathbb{Z}_p)_{S,1} \supseteq X(\mathbb{Z}_p)_{S,2} \supseteq \dots$$

all containing $X(\mathbb{Z}_S)$, as in the projective higher genus case.

Kim (2005): $\#X(\mathbb{Z}_p)_{S,n} < \infty$ for $n \gg 0$

Dan-Cohen, Wewers, Brown, Corwin: motivic variant of Chabauty–Kim method

Refined Chabauty–Kim

Betts–Dogra (2020): refined Chabauty–Kim loci

$$X(\mathbb{Z}_p) \supseteq X(\mathbb{Z}_p)_{S,1}^{\min} \supseteq X(\mathbb{Z}_p)_{S,2}^{\min} \supseteq \dots$$

Idea: partition S-integral points by their reductions modulo primes $\ell \in S$

$$\mathrm{red}_\ell \colon X(\mathbb{Z}_S) \subseteq \mathbb{P}^1(\mathbb{Z}_S) = \mathbb{P}^1(\mathbb{Z}) \twoheadrightarrow \mathbb{P}^1(\mathbb{F}_\ell)$$

Refined Kim's Conjecture

$X(\mathbb{Z}_p)_{S,n}^{\min} = X(\mathbb{Z}_S)$ for $n \gg 0$

- proved for $S = \{2\}$ and all odd p in depth $\max(1, p-3)$ (Betts–Kumpitsch–L. 2023)
- proved for $S = \{2, q\}$ and $p = 3$ in depth 2 when $q = 2^n \pm 1 > 3$ Fermat or Mersenne (Best–Betts–Kumpitsch–L.–McAndrew–Qian–Studnia–Xu 2024)
- case $S = \{2, 3\}$: depth 2 does not suffice $\to$ go to depth 4 (later)

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$$X(\mathbb{Z}_S)_{(1,0)} := \{x \in X(\mathbb{Z}_S) : \mathrm{red}_2(x) \in X \cup \{1\},\ \mathrm{red}_q(x) \in X \cup \{0\}\}$$

and associated refined Chabauty–Kim loci $X(\mathbb{Z}_p)^{(1,0)}_{S,n}$.

Theorem (BBKLMQSX)

The depth 2 locus $X(\mathbb{Z}_p)^{(1,0)}_{\{2,q\},2}$ is defined inside $X(\mathbb{Z}_p)$ by

$$\mathrm{Li}_2(z) - a \log(z) \mathrm{Li}_1(z) = 0$$

for some computable p-adic constant $a = a(q) \in \mathbb{Q}_p$.

Here, log is the p-adic logarithm and Li_m is the p-adic polylogarithm

$$\mathrm{Li}_m(z) = \int_0^z \frac{dx}{x} \cdots \frac{dx}{x} \frac{dx}{1-x} \qquad (m\text{-fold iterated integral})$$

Depth 2 loci

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Here, log is the p-adic logarithm and Li_m is the p-adic polylogarithm

$$\mathrm{Li}_m(z) = \int_0^z \frac{dx}{x} \cdots \frac{dx}{x} \frac{dx}{1-x} \qquad (m\text{-fold iterated integral})$$

Computing depth 2 loci

L.: Sage code for computing $X(\mathbb{Z}_p)^{(1,0)}_{\{2,q\},2}$ for arbitrary p and q

→ https://github.com/martinluedtke/RefinedCK

Example: $S = \{2,3\}$, $p = 5$. Have $X(\mathbb{Z}[1/6])_{(1,0)} = \{-3,-1,3,9\}$.

```
p = 5; q = 3
a = -Qp(p)(3).polylog(2)
CK_depth_2_locus(p,q,10,a)
```

Output:

```
[2 + O(5^9),
 2 + 4*5 + 4*5^2 + 4*5^3 + 4*5^4 + 4*5^5 + 4*5^6 + 4*5^7 + 4*5^8 + O(5^9),
 3 + O(5^6),
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```

Analysing the size of depth 2 loci

How does the size of $X(\mathbb{Z}_p)^{(1,0)}_{\{2,3\},2}$ vary with the choice of auxiliary prime p?

p	5	7	11	13	17	19	23	29	31	...	1091	1093	1097	...
$\#X(\mathbb{Z}_p)^{(1,0)}_{\{2,3\},2}$	6	8	18	16	22	20	20	26	36	...	1076	2154	1078	...

Observations:

- size is even
- 0 or 2 points in each residue disc
- almost always of size $\approx p$, but for $p \in \{1093, 3511\}$ of size $\approx 2p$

Can explain this heuristically. Related to 1093 and 3511 being (the only known) Wieferich primes, i.e., primes with $2^{p-1} \equiv 1 \bmod p^2$.

Similar observations for $S = \{2, q\}$ with q different from 3.

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Ingredients for computing depth 2 loci

$$\boxed{\mathrm{Li}_2(z) - a(q)\log(z)\,\mathrm{Li}_1(z) = 0}$$

How to compute the zero locus?

1. Compute the p-adic constant $a(q)$ using modified algorithm of Dan-Cohen–Wewers
   → https://github.com/martinluedtke/dcw_coefficients
2. Compute power series for polylogarithms on residue discs around roots of unity ζ

$$\mathrm{Li}_m(\zeta + pt) = \sum_{k=1}^{\infty} a_{m,k} t^k$$

   → Besser–de Jeu's "$\mathrm{Li}^{(p)}$-service" paper
3. Implemented Hensel's Lemma for finding roots of p-adic power series with correct precision: function `Zproots`
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Depth 4 loci

Adapting work of Corwin and Dan-Cohen to the refined setting, we derive a new function for the refined depth 4 locus in the case $S = \{2, 3\}$:

Theorem (L. 2025)

Let $S = \{2, 3\}$ and $p \notin S$. Any point z in the refined Chabauty–Kim locus $X(\mathbb{Z}_p)^{(1,0)}_{\{2,3\},4}$ satisfies, in addition to the depth 2 equation, the equation

$$\det \begin{pmatrix} \mathrm{Li}_4(z) & \log(z)\,\mathrm{Li}_3(z) & \log(z)^3\,\mathrm{Li}_1(z) \\ \mathrm{Li}_4(3) & \log(3)\,\mathrm{Li}_3(3) & \log(3)^3\,\mathrm{Li}_1(3) \\ \mathrm{Li}_4(9) & \log(9)\,\mathrm{Li}_3(9) & \log(9)^3\,\mathrm{Li}_1(9) \end{pmatrix} = 0.$$

Also have a depth 4 equation for general $S = \{2, q\}$ but it is less explicit.

Computing depth 4 loci for $S = \{2, 3\}$

Use the new equation to compute depth 4 locus $X(\mathbb{Z}_p)^{(1,0)}_{\{2,3\},4}$:

```
p = 5; q = 3; N = 10
coeffs = Z_one_sixth_coeffs(p,N)
CK_depth_4_locus(p,q,N,coeffs)
```

Output:

```
[2 + 4*5 + 4*5^2 + 4*5^3 + 4*5^4 + 4*5^5 + 4*5^6 + 4*5^7 + 4*5^8 + O(5^9),
 3 + O(5^6),
 4 + 4*5 + 4*5^2 + 4*5^3 + 4*5^4 + 4*5^5 + 4*5^6 + 4*5^7 + 4*5^8 + O(5^9),
 4 + 5 + O(5^9)]
```

This is $\{-3, 3, -1, 9\} = X(\mathbb{Z}[1/6])_{(1,0)}$, extra points are eliminated.

$\Rightarrow$ Refined Kim's Conjecture holds for $S = \{2, 3\}$ and $p = 5$

Verifying Kim's Conjecture

I computed depth 4 loci for $S = \{2, 3\}$ for many primes p:

Theorem (L. 2025)

The Refined Kim's Conjecture holds for $S = \{2, 3\}$ and all primes $3 < p < 10{,}000$.

Summary

- Sage code to compute depth 2 Chabauty–Kim loci for $S = \{2, q\}$ and all p
- Analysed sizes of those loci, explained numerical observations
- Derived functions vanishing on depth 4 loci
- Verified Kim's Conjecture for $S = \{2, 3\}$ and $p < 10{,}000$

References

[BBK+24] A. Best, L. A. Betts, T. Kumpitsch, M. Lüdtke, A. McAndrew, L. Qian, E. Studnia, and Y. Xu. "Refined Selmer equations for the thrice-punctured line in depth two". In: *Math. Comp.* 93 (2024), pp. 1497–1527. DOI: 10.1090/mcom/3898.

[BD20] L. A. Betts and N. Dogra. *The local theory of unipotent Kummer maps and refined Selmer schemes*. 2020. arXiv: 1909.05734v2 [math.NT].

[BKL23] L. A. Betts, T. Kumpitsch, and M. Lüdtke. *Chabauty–Kim and the Section Conjecture for locally geometric sections*. 2023. arXiv: 2305.09462v1 [math.NT].

[Bro17] F. Brown. *Integral points on curves, the unit equation, and motivic periods*. 2017. arXiv: 1704.00555v1 [math.NT].

[CD20] D. Corwin and I. Dan-Cohen. "The polylog quotient and the Goncharov quotient in computational Chabauty–Kim theory I". In: *Int. J. Number Theory* 16 (2020), pp. 1859–1905.

[DW15] I. Dan-Cohen and S. Wewers. "Explicit Chabauty–Kim theory for the thrice punctured line in depth 2". In: *Proc. Lond. Math. Soc.* 110.1 (2015), pp. 133–171.

[Kim05] M. Kim. "The motivic fundamental group of $\mathrm{P}^1 \smallsetminus \{0, 1, \infty\}$ and the theorem of Siegel". In: *Invent. Math.* 161.3 (2005), pp. 629–656.

[Lüd25] M. Lüdtke. "Refined Chabauty–Kim computations for the thrice-punctured line over $\mathbb{Z}[1/6]$". In: *Res. Number Theory* 11 (2025). DOI: 10.1007/s40993-024-00597-4.

Thanks for listening...

Hereweg Groningen in 1906.
Source: https://www.groningerarchieven.nl

(18) In the summer of 1936 at Groningen in the Netherlands, when I was still working at the University there, a bicycle rider ran into me. As a consequence, the tuberculosis in my right knee bone, which had been dormant for many years, flared up again. It therefore became necessary to undergo several bone operations in 1936 and 1937. This was naturally a very painful period and I was given many morphine injections, although my doctor warned me against their danger.

After a further operation the pains and hence also the injections finally stopped. Then I tried to convince myself that the drug had not damaged my brain by studying the problem of the possible transcendency of the decimal fraction

$$D = 0.123456789101112...$$

in which the successive integers are written one after the other. I found that I could still do mathematics and succeeded in proving the transcendency of both D and of infinitely many more general decimal fractions.

From: K. Mahler, *Fifty years as a mathematician*

...and cycle responsibly in Groningen!