Oberwolfach workshop: Homotopic & Geometric Galois Theory 9 March 2021

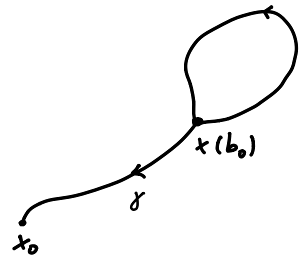
The p-adic section conjecture for localisations of curves

§1. The section conjecture

$$\begin{array}{ll} X & \text{continuous map of top. spaces} \\ f\big\downarrow & \\ \mathcal{B} & \text{base points } x_0 \mapsto b_0 \end{array}$$

Consider a section $x : \mathcal{B} \longrightarrow X$ of f.

Get $x_* : \pi_1(\mathcal{B}, b_0) \longrightarrow \pi_1(X, x(b_0))$ by functoriality.



Assume that the fibre $f^{-1}(b_0)$ is path-connected.

Choose path $\gamma : x(b_0) \rightsquigarrow x_0$

$\rightsquigarrow$ get $\gamma(-)\gamma^{-1} : \pi_1(X, x(b_0)) \xrightarrow{\sim} \pi_1(X, x_0)$

$\rightsquigarrow$ section $s_x : \pi_1(B, b_0) \xrightarrow{x_*} \pi_1(X, x(b_0)) \xrightarrow{\gamma(-)\gamma^{-1}} \pi_1(X, x_0)$ of f_*

Different choice γ'

$\rightsquigarrow$ loop $\gamma'\gamma^{-1} \in \pi_1(X_{b_0}, x_0)$ in the fibre $X_{b_0} = f^{-1}(b_0)$

$\rightsquigarrow$ s_x well-defined up to $\pi_1(X_{b_0}, x_0)$ – conjugacy

<u>To summarise:</u>

$$\{\text{sections of } f\} \longrightarrow \left\{\begin{matrix}\pi_1(X_{b_0}, x_0)\text{-conjugacy classes} \\ \text{of sections of } f_*\end{matrix}\right\}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ B \arrow[u, bend right, dashed, "x"'] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \pi_1(X, x_0) \arrow[d, "f_*"'] \\ \pi_1(B, b_0) \arrow[u, bend right, dashed, "s_x"'] \end{tikzcd}$$

<u>In arithmetic geometry</u>

k field, $\bar{k}/k$ alg. closure

X/k geom. connected, $\bar{x}_0 \in X(\bar{k})$

Apply the construction above to $f : X \longrightarrow \mathrm{Spec}(k)$ using étale fundamental groups.

Have $\pi_1(\mathrm{Spec}(k), \mathrm{Spec}(\bar{k})) = \mathrm{Gal}(\bar{k}/k) =: G_k$.

Get map

$$X(k) \longrightarrow \left\{\begin{matrix}\pi_1(X_{\bar{k}}, \bar{x}_0)\text{- conjugacy classes} \\ \text{of sections of } \pi_1(X, \bar{x}_0) \to G_k\end{matrix}\right\}. \qquad (*)$$

$$\begin{tikzcd} X \arrow[d] \\ \mathrm{Spec}(k) \arrow[u, bend right, dashed] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \pi_1(X, \bar{x}_0) \arrow[d] \\ G_k \arrow[u, bend right, dashed] \end{tikzcd}$$

> <u>Section conjecture:</u> (Grothendieck 1983)
>
> Let k be f.g. over $\mathbb{Q}$, and let X/k be a proper hyperbolic curve.
> Then $(*)$ is a bijection.

Injectivity already known by Grothendieck.
(Consequence of Mordell-Weil Theorem)

Variant for open curves

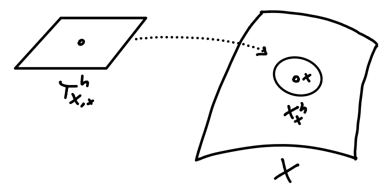
$\mathrm{char}(k) = 0$, X/k smooth curve, $X \subseteq \overline{X}$ compactification

Points at infinity induce sections as well:

Let $x \in (\overline{X} \setminus X)(k)$ be a cusp (or point at infinity).

$X_x^h := \mathrm{Spec}(\mathrm{Frac}(\mathcal{O}^h_{\overline{X},x}))$ henselisation of X at x

$=$ algebraic analogue of punctured disc



$T^\circ_{X,x}/k$ punctured tangent scheme at x

$\cong \mathbb{G}_m$ (non-canonically)

Deligne's theory of tangential base points

$\Rightarrow \pi_1(T^\circ_{X,x}) \simeq \pi_1(X_x^h)$

Consequence: all nonzero k-rational tangent vectors $v \in T^\circ_{X,x}(k)$ induce sections of $\pi_1(X) \longrightarrow G_k$:

$$\begin{tikzcd} \pi_1(T^\circ_{X,x}) \arrow[d] \\ \pi_1(X) \arrow[d] \\ \mathrm{Spec}(k) \arrow[uu, dashed, bend right, "s_v"'] \end{tikzcd}$$

"cuspidal section"

Section Conjecture for open curves:

Let X/k be a smooth, hyperbolic curve.
Then every section of $\pi_1(X, x_0) \twoheadrightarrow G_k$ is induced by a unique k-point of $\overline{X}$.

Birational variant

X is hyperbolic if the Euler characteristic is negative:

$$\chi(X) = 2 - 2g - r < 0$$

where

g = genus of X
$r := \#(\bar{X} \setminus X)(\bar{k})$ number of cusps

Thus:

"more cusps $\Rightarrow$ more hyperbolic"

Removing all closed points results in the generic point η_X.

$\eta_X = \mathrm{Spec}(K)$, K = function field of X

$\pi_1(\eta_X) = \mathrm{Gal}(\bar{K}/K) =: G_K$

> **Birational Section Conjecture:**
>
> Let X/k be a smooth, proper curve, K/k the function field.
>
> Every section of $G_K \twoheadrightarrow G_k$ is induced by a unique k-rational point of X.

§2. The p-adic section conjecture for localisations of curves

From now on: k = finite extension of $\mathbb{Q}_p$

Section conjecture for proper or open hyperbolic curves over k is open.

Birational p-adic section conjecture

Theorem: (p-adic Birational SC, Koenigsmann 2005)
The Birational SC holds for k a finite extension of $\mathbb{Q}_p$.

Proof uses model theory of p-adically closed fields.

Different proof was given by Pop in 2010.

Main technical input:

Theorem: (Pop 1988)
Local-to-global principle for Brauer groups of fields M/k of transcendence degree 1:

$$\mathrm{Br}(M) \hookrightarrow \prod_w \mathrm{Br}(M_w^h) \quad \text{is injective}$$

w valuation on M extending p-adic valuation on k,
M_w^h henselisation of M at w

Rough sketch of Pop's proof: Given section $\begin{tikzcd} G_K \arrow[r, two heads] & G_k \arrow[l, bend right, dashed, "s"'] \end{tikzcd}$.

1. $\mathrm{im}(s) \subseteq G_K$ corresponds to Galois extension M/K with $\mathrm{Gal}(\bar{K}/M) \xrightarrow{\sim} \mathrm{Gal}(\bar{k}/k)$

2. $\mathrm{Br}(k) \longrightarrow \mathrm{Br}(M)$ injective.
   Let $\alpha \in \mathrm{Br}(k)$ with $\mathrm{inv}(\alpha) = \frac{1}{p} \bmod \mathbb{Z}$.
   $\alpha \neq 0 \Rightarrow \alpha|_M \neq 0$

3. Local-to-global principle
   $\Rightarrow \exists$ valuation w on M s.t. $\alpha|_{M_w^h} \neq 0$

4. $w|K$ is the valuation defined by a closed point x of X,
   s is induced by this point x

In fact, Pop proves a "minimalistic" variant which uses only the maximal $\mathbb{Z}/p\mathbb{Z}$ - metabelian quotient of $\pi_1(X)$.

Localisations of curves

X/k smooth, proper curve

$S \subseteq X_{cl}$ arbitrary set of closed points

Def: localisation of X at S:

$$X_S := \bigcap_{U \supseteq S} U, \qquad U \subseteq X \text{ dense open}$$

Ex:
- $S = X_{cl}$: $X_{X_{cl}} = X$ the whole curve
- $S = \emptyset$: $X_\emptyset = \eta_X$ the generic point
- $S = \{x\}$: $X_x = \mathrm{Spec}(\mathcal{O}_{X,x})$

In general:

$$\{\eta_X\} \subseteq X_S \subseteq X$$

interpolation between generic point and whole curve

$$G_K \twoheadrightarrow \pi_1(X_S) \twoheadrightarrow \pi_1(X)$$

Section conjecture for the localisation X_S:

Every section of $\pi_1(X_S) \twoheadrightarrow G_k$ is induced by a unique k-rational point of X.

We identify conditions on X and $S \subseteq X_{cl}$ which ensure that Pop's proof generalises from η_X to X_S.

We verify the conditions in some cases:

Theorem A: (L. 2020)

Assume that

(a) S is at most countable, **or**

(b) X is defined over a subfield $k_0 \subseteq k$ and $S = \{\text{transcendental points over } k_0\} \cup (\text{finite})$.

Then X_S satisfies the section conjecture.

§3. The liftable section conjecture

Notation: Π profinite group

$\Pi' := \Pi^{ab} \otimes \mathbb{Z}/p\mathbb{Z}$ maximal p-elementary abelian quotient

$\Pi'' :=$ maximal $\mathbb{Z}/p\mathbb{Z}$-metabelian quotient

Def: $\Pi \twoheadrightarrow G$ surjection of profinite groups.

section s' of $\Pi' \twoheadrightarrow G'$ is liftable if $\exists\, s''$ as follows:

$$\begin{tikzcd} \Pi'' \arrow[r, two heads] \arrow[d, two heads] & G'' \arrow[l, bend right, dashed, "s''"'] \arrow[d, two heads] \\ \Pi' \arrow[r] & G' \arrow[l, bend right, "s'"'] \end{tikzcd}$$

Def: We say that X_S/k satisfies the liftable section conjecture if every liftable section s' of $\pi_1(X_S)' \longrightarrow G_k'$ is induced by a unique k-rational point of X.

Theorem: Assume that every geometrically connected finite étale cover of X_S satisfies the liftable section conjecture.

Then X_S satisfies the section conjecture.

Good localisations

$k/\mathbb{Q}_p$ finite, $\mu_p \subseteq k$

X/k smooth, proper curve

K the function field of X

$S \subseteq X_{cl}$ set of closed points

Def: X_S/k is a good localisation if the following four conditions are satisfied:

(Sep) For all $x \neq y$ in $X(k)$, the map

$$\mathcal{O}(X_{S \cup \{x,y\}})^\times \longrightarrow k^\times / k^{\times p}, \qquad f \longmapsto \frac{f(x)}{f(y)}$$

is nontrivial.

(Pic) Every geometrically connected, finite p-elementary abelian cover $W \longrightarrow X_S$ satisfies $\mathrm{Pic}(W)/p = 0$.

(Rat) For all non-rational closed points $x \in X_{cl}$ with $p \nmid \deg(x)$, the map

$$\mathcal{O}(X_{S \cup \{x\}})^\times \longrightarrow \kappa(x)^\times / k^\times \kappa(x)^{\times p}, \qquad f \longmapsto f(x)$$

is nontrivial.

(Fin) For every rank 1 valuation w on K extending the p-adic valuation on k, the map

$$\mathcal{O}(X_S)^\times \longrightarrow (K_w^h)^\times / (K_w^h)^{\times p}$$

has finite cokernel.

Theorem B: (L. 2020)

Good localisations satisfy the liftable section conjecture.

<u>Example:</u> $\eta_X = X_\emptyset$ is a good localisation
$\Rightarrow$ recover birational p-adic SC (liftable + full)

For (Rat):

<u>Lemma:</u> $k \subsetneq \ell$ nontrivial extension
$\Rightarrow k^\times/k^{\times p} \longrightarrow \ell^\times/\ell^{\times p}$ not surjective

<u>Pf:</u> $\dim_{\mathbb{F}_p}\left(k^\times/k^{\times p}\right) < \dim_{\mathbb{F}_p}\left(\ell^\times/\ell^{\times p}\right)$.

<u>Sketch of proof of Theorem B:</u>

Let X_S be a good localisation.
Let $s': G_k' \longrightarrow \pi_1(X_S)$ be a liftable section.
Let $W := W[s'] \longrightarrow X_S$ be the p-elementary abelian cover corresponding to $\mathrm{im}(s') \subseteq \pi_1(X_S)'$, let $M := M[s']$ be its function field.

<u>Step 1:</u> $\mathrm{Br}(k)[p] \longrightarrow \mathrm{Br}(M)[p]$ is injective.

- use liftability to show $H^2(G_k, \mu_p) \longrightarrow H^2(\pi_1(W), \mu_p)$ injective
- comparison of group cohomology and étale cohomology:
  $\Rightarrow H^2(\pi_1(W), \mu_p) \hookrightarrow H^2(W, \mu_p)$ injective
- use condition (Pic) and Kummer sequence

$$0 \longrightarrow \underbrace{\mathrm{Pic}(W)/p}_{=0} \longrightarrow H^2(W, \mu_p) \longrightarrow \mathrm{Br}(W)[p] \longrightarrow 0$$

- use $\mathrm{Br}(W) \subseteq \mathrm{Br}(M)$ (Grothendieck purity)

Byproduct of Step 1:

<u>Theorem:</u> (L. 2020)
Assume $\mathrm{genus}(X) > 0$. If there exists $S \subseteq X_{cl}$ s.t. X_S satisfies (Pic) and $\pi_1(X_S) \to G_k$ admits a section, then

$$\mathrm{index}(X) \overset{\mathrm{def}}{=} \gcd\{[k(x):k] \mid x \in X_{cl}\} = 1.$$

<u>Step 2:</u> Let $\alpha \in \mathrm{Br}(k)$ the class with $\mathrm{inv}(\alpha) = \frac{1}{p} \pmod{\mathbb{Z}}$.
Have $\alpha|_M \neq 0$ by Step 1.
Pop's Local-to-global principle

$\Rightarrow$ $\exists$ valuation w on M of rank 1 with $w|_k \in \{p\text{-adic, trivial}\}$

s.t. $\alpha|_{M_w^h} \neq 0$.

<u>Step 3</u>: rule out positive residue characteristic of w:

Assume $\mathrm{char}(\kappa(w)) > 0$. Look at extension M_w^h / K_w^h.

Condition (Fin) implies: M_w^h is cofinite in the maximal p-elementary abelian extension of K_w^h.

Analyse p-elementary abelian extensions of mixed char. henselian fields

$\Rightarrow$ M_w^h too large for Brauer class α to survive $\lightning$

So $w|_k$ = trivial.

$\Rightarrow$ $w|_K = v_x$ valuation of a closed point $x \in X_{cl}$.

<u>Step 4:</u> Condition (Rat) implies that x is k-rational.

Condition (Sep) implies uniqueness statement.

Variant of liftable section conjecture without p-th roots of unity:

Let ℓ/k be finite Galois.

<u>Def:</u> Say a section $s' : \mathrm{Gal}(\ell'/k) \longrightarrow \mathrm{Gal}((X_S \otimes \ell)'/X_S)$ is *liftable* if it admits s'' as follows:

$$\begin{tikzcd}
\mathrm{Gal}((X_S\otimes\ell)''/X_S) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Gal}(\ell''/k) \arrow[l, dashed, bend right, "s''"'] \arrow[d, two heads] \\
\mathrm{Gal}((X_S\otimes\ell)'/X_S) \arrow[r, two heads] & \mathrm{Gal}(\ell'/k) \arrow[l, bend right, "s'"']
\end{tikzcd}$$

<u>Theorem:</u> Assume that $X_S \otimes \ell$ satisfies the liftable SC.

Let $s' : \mathrm{Gal}(\ell'/k) \longrightarrow \mathrm{Gal}((X_S \otimes \ell)'/X_S)$ be a liftable section.

Then there exists a unique k-rational point $x \in X(k)$

s.t. $s'|_{\mathrm{Gal}(\ell'/\ell)}$ lies over $x \otimes \ell$.