Mixed Tate Selmer schemes beyond the polylog quotient

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1. Chabauty–Kim functions for the thrice-punctured line

Siegel's theorem

Setup:

- S: finite set of primes
- $\mathbb{Z}_S = \mathcal{O}(\mathrm{Spec}(\mathbb{Z}) \smallsetminus S)$: ring of S-integers
- $X = \mathbb{P}^1_{\mathbb{Z}_S} \smallsetminus \{0, 1, \infty\}$: thrice-punctured line

Theorem (Siegel 1929)

$X(\mathbb{Z}_S)$ is finite.

Chabauty–Kim loci

Chabauty–Kim theory is a p-adic approach to computing $X(\mathbb{Z}_S)$.

For $p \notin S$, we have a Chabauty–Kim locus

$$X(\mathbb{Z}_p)_{S,\infty} \subseteq X(\mathbb{Z}_p).$$

- It is defined as the common vanishing set of a collection of Coleman functions on $X(\mathbb{Z}_p)$.
- It contains $X(\mathbb{Z}_S)$.
- It is finite (Kim[1], 2005).
- It is conjectured that $X(\mathbb{Z}_p)_{S,\infty}$ is *exactly* the set of S-integral points $X(\mathbb{Z}_S)$.

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[1] *The motivic fundamental group of* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ *and the theorem of Siegel*

Explicit equations

Finding explicit Coleman functions defining $X(\mathbb{Z}_p)_{S,\infty}$ is difficult in practice and has been achieved only for small S.

Theorem (Dan-Cohen–Wewers[2], 2013)

Let $S = \{2\}$. The following functions vanish on $X(\mathbb{Z}_p)_{\{2\},\infty}$:

$$2\,\mathrm{Li}_2(z) - \log(z)\,\mathrm{Li}_1(z)$$

$$\log(2)\zeta(3)\,\mathrm{Li}_4(z) - \frac{8}{7}\,\mathrm{Li}_4(2)\log(z)\,\mathrm{Li}_3(z) - \frac{1}{24}\left(\log(2)\zeta(3) - \frac{32}{7}\,\mathrm{Li}_4(2)\right)\log(z)^3\,\mathrm{Li}_1(z)$$

[2] *Mixed Tate motives and the unit equation*

Refined Chabauty–Kim loci

Other results use the refined Chabauty–Kim method.

For $\Sigma = (\Sigma_\ell)_{\ell \in S}$ with $\Sigma_\ell \in \{0, 1, \infty\}$, define

$X(\mathbb{Z}_S)_\Sigma \coloneqq \{z \in X(\mathbb{Z}_S) : (z \bmod \ell) \in X(\mathbb{F}_\ell) \cup \{\Sigma_\ell\} \text{ for all } \ell \in S\}.$

Note that $X(\mathbb{Z}_S) = \bigcup_\Sigma X(\mathbb{Z}_S)_\Sigma$.

The refined method produces partial Chabauty–Kim loci $X(\mathbb{Z}_p)^\Sigma_{S,\infty}$ which are conjecturally equal to $X(\mathbb{Z}_S)_\Sigma$.

Explicit refined equations for $S = \{2\}$

Theorem (Betts–Kumpitsch–L.[3], 2023)

Let $S = \{2\}$. The following functions vanish on $X(\mathbb{Z}_p)^{(1)}_{\{2\},\infty}$:

$$\log(z) \quad \textit{and} \quad \mathrm{Li}_k(z) \textit{ for } k \geq 2 \textit{ even.}$$

Remark: In this case we can show the conjectured equality

$$X(\mathbb{Z}_p)^{(1)}_{\{2\},\infty} = \{-1\} = X(\mathbb{Z}[1/2])_{(1)}$$

[3] *Chabauty–Kim and the Section Conjecture for locally geometric sections*

Explicit refined equations for $S = \{2, q\}$

Theorem (Best–Betts–Kumpitsch–L.–McAndrew–Qian–Studnia–Xu[4], 2021)

Let $S = \{2, q\}$ for some odd prime q. The following function vanishes on $X(\mathbb{Z}_p)^{(1,0)}_{\{2,q\},\infty}$:

$$\log(2)\log(q)\,\mathrm{Li}_2(z) - a_{q,2}\log(z)\,\mathrm{Li}_1(z).$$

Here, $a_{q,2}$ is a certain computable p-adic constant.

[4] *Refined Selmer equations for the thrice-punctured line in depth two*

Polylogarithms versus multiple polylogarithms

Note that all of these functions are polynomials in polylogarithms: $\log, \mathrm{Li}_1, \mathrm{Li}_2, \ldots$. These are given by the iterated Coleman integrals

$$\log(z) = \int_0^z \frac{\mathrm{d}z}{z}, \quad \mathrm{Li}_k(z) = \int_0^z \underbrace{\frac{\mathrm{d}z}{z} \cdots \frac{\mathrm{d}z}{z} \frac{\mathrm{d}z}{1-z}}_{k}.$$

This is because all calculations so far have been restricted to working with the *polylogarithmic quotient* of the fundamental group of X. There are however a lot more Coleman functions on $X(\mathbb{Z}_p)$. Specifically, for any tuple $(k_1, \ldots, k_r)$, $k_i \geq 1$, we have the multiple polylogarithm

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \int_0^z \underbrace{\frac{\mathrm{d}z}{z} \cdots \frac{\mathrm{d}z}{z} \frac{\mathrm{d}z}{1-z}}_{k_1} \cdots \underbrace{\frac{\mathrm{d}z}{z} \cdots \frac{\mathrm{d}z}{z} \frac{\mathrm{d}z}{1-z}}_{k_r}.$$

Polylogarithms versus multiple polylogarithms

Question

Can we remove the restriction to the polylogarithmic quotient and find new Coleman functions defining Chabauty–Kim loci?

Answer

Yes. But the calculations become more complicated.

The motivic Selmer scheme for the fundamental group quotient Π is a space of $\mathbb{G}_m$-equivariant cocycles

$$\mathrm{Z}^1(U_S, \Pi)^{\mathbb{G}_m}.$$

The action of U_S on the polylogarithmic fundamental group is trivial, so that $\mathrm{Z}^1 = \mathrm{Hom}$. In general, we need to understand the motivic Galois action.

Explicit non-polylogarithmic functions

Theorem (Corwin–Dan-Cohen–L., 2023)

Let $S = \{2\}$. The following functions vanish on $X(\mathbb{Z}_p)_{\{2\},\infty}$:

- $2\,\mathrm{Li}_2(z) - \log(z)\,\mathrm{Li}_1(z)$,
- $\log(2)\zeta(3)\,\mathrm{Li}_4(z) - \frac{8}{7}\,\mathrm{Li}_4(2)\log(z)\,\mathrm{Li}_3(z) - \frac{1}{24}\left(\log(2)\zeta(3) - \frac{32}{7}\,\mathrm{Li}_4(2)\right)\log(z)^3\,\mathrm{Li}_1(z)$,
- $\log(2)\zeta(3)\left(\mathrm{Li}_{3,1}(z) + \frac{1}{8}\log(z)^2\,\mathrm{Li}_1(z)^2\right) - \frac{8}{7}\,\mathrm{Li}_4(2)\log(z)\,\mathrm{Li}_{2,1}(z) - \frac{4}{7}\,\mathrm{Li}_{3,1}(-1)\,\mathrm{Li}_1(z)\,\mathrm{Li}_3(z) + \zeta(3)\frac{4}{7}\,\mathrm{Li}_{3,1}(-1)\,\mathrm{Li}_1(z)$,
- $\log(2)\zeta(3)\,\mathrm{Li}_{2,1,1}(z) - \frac{4}{7}\,\mathrm{Li}_{3,1}(-1)\,\mathrm{Li}_1(z)\,\mathrm{Li}_{2,1}(z) - \frac{1}{24}\left(\log(2)\zeta(3) - \frac{32}{7}\,\mathrm{Li}_4(2)\right)\log(z)\,\mathrm{Li}_1(z)^3 + \frac{4}{7}\,\mathrm{Li}_{3,1}(-1)\zeta(3)\,\mathrm{Li}_4(-1)\,\mathrm{Li}_1(z)$.

Explicit non-polylogarithmic functions

Theorem (Corwin–Dan-Cohen–L., 2023)

Let $S = \{2\}$. The following functions vanish on $X(\mathbb{Z}_p)^{(1)}_{\{2\},\infty}$:

- $\log(z)$,
- $\mathrm{Li}_2(z)$,
- $\mathrm{Li}_4(z)$,
- $\zeta(3)\log(2)\,\mathrm{Li}_{3,1}(z) - \frac{4}{7}\,\mathrm{Li}_{3,1}(-1)\,\mathrm{Li}_1(z)\big(\mathrm{Li}_3(z) - \zeta(3)\big)$,
- $\zeta(3)\log(2)\,\mathrm{Li}_{2,1,1}(z) - \frac{4}{7}\,\mathrm{Li}_{3,1}(-1)\,\mathrm{Li}_1(z)\big(\mathrm{Li}_{2,1}(z) - \zeta(3)\big)$.

Explicit non-polylogarithmic functions

Theorem (Corwin–Dan-Cohen–L., 2023)

Let $S = \{2, q\}$ for an odd prime q. The following functions vanish on $X(\mathbb{Z}_p)^{(1,0)}_{\{2,q\},\infty}$:

- $a_{\tau_2} a_{\tau_q} \operatorname{Li}_2(z) - a_{\tau_q\tau_2} \log(z) \operatorname{Li}_1(z)$
- $a_{\sigma_3} a_{\tau_2} a_{\tau_q}^3 \operatorname{Li}_4(z) - a_{\tau_2} a_{\tau_q}^2 a_{\tau_q\sigma_3} \log(z) \operatorname{Li}_3(z)$
  $- \left(a_{\sigma_3} a_{\tau_q\tau_q\tau_q\tau_2} a_{\tau_q\sigma_3} - a_{\tau_q\tau_q\tau_2}\right) \log(z)^3 \operatorname{Li}_1(z),$
- $a_{\tau_2}^2 a_{\tau_q}^2 a_{\sigma_3} \operatorname{Li}_{3,1}(z) - a_{\tau_2} a_{\tau_q}^2 a_{\sigma_3\tau_2} \operatorname{Li}_1(z)(\operatorname{Li}_3(z) - a_{\sigma_3})$
  $- a_{\tau_2}^2 a_{\tau_q} a_{\tau_q\sigma_3} \log(z) \operatorname{Li}_{2,1}(z)$
  $- (a_{\sigma_3} a_{\tau_q\tau_q\tau_2\tau_2} - a_{\sigma_3\tau_2} a_{\tau_q\tau_q\tau_2} - a_{\tau_q\sigma_3} a_{\tau_q\tau_2\tau_2}) \log(z)^2 \operatorname{Li}_1(z)^2,$
- $a_{\sigma_3} a_{\tau_2}^3 a_{\tau_q} \operatorname{Li}_{2,1,1}(z) - a_{\sigma_3\tau_2} a_{\tau_2}^2 a_{\tau_q} \operatorname{Li}_1(z)(\operatorname{Li}_{2,1}(z) - a_{\sigma_3})$
  $- (a_{\sigma_3} a_{\tau_q\tau_2\tau_2\tau_2} - a_{\sigma_3\tau_2} a_{\tau_q\tau_2\tau_2}) \log(z) \operatorname{Li}_1(z)^3.$

Here, the a_u are certain (hard to compute) p-adic constants.

Motivic Chabauty–Kim method

These calculations are carried out using the motivic Selmer scheme $\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X)$ for some quotient Π of the *motivic fundamental group* of X. It sits in a motivic Chabauty–Kim diagram as follows:

$$\begin{array}{ccc} X(\mathbb{Z}_S) & \hookrightarrow & X(\mathbb{Z}_p) \\ \downarrow j_S & & \downarrow j_p \\ \mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X)(\mathbb{Q}) & \xrightarrow{\mathrm{loc}_p} & \Pi^{\mathrm{dR}}(\mathbb{Q}_p) \end{array}$$

Strategy:

1. Show that $\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X)$ is an affine space over $\mathbb{Q}$ with specified coordinates.
2. Describe the localisation map loc_p using these coordinates.
3. Find functions f on $\Pi^{\mathrm{dR}}_{\mathbb{Q}_p}$ which vanish on the image of loc_p.
4. The pullbacks $f \circ j_p$ are Coleman functions vanishing on $X(\mathbb{Z}_p)_{S,\infty}$.

2. The motivic Selmer scheme

Algebraic geometry in a Tannakian category

Let k be a field and $\mathcal{T} = (\mathcal{T}, \otimes, 1)$ a k-linear Tannakian category.

A ring in $\mathcal{T}$ is an ind-object A of $\mathcal{T}$ with a unit $1 \to A$ and multiplication map $A \otimes A \to A$ satisfying unit, associativity and commutativity axioms.

An affine scheme in $\mathcal{T}$ is $\mathrm{Spec}(A)$ for a ring A in $\mathcal{T}$.

Similar definitions for affine group schemes and torsors in $\mathcal{T}$.

Example

If $\mathcal{T}$ is the category of linear representations of a pro-algebraic group G/k, an affine scheme in $\mathcal{T}$ is an affine k-scheme X with a G-action.

Mixed Tate motives

Let $\mathsf{MT}(\mathbb{Z}_S, \mathbb{Q})$ be the category of mixed Tate motives over $\mathbb{Z}_S$ with $\mathbb{Q}$-coefficients, as constructed by Deligne–Goncharov.[5]

- $\mathsf{MT}(\mathbb{Z}_S, \mathbb{Q})$ is a $\mathbb{Q}$-linear Tannakian category.
- distinguished object: $\mathbb{Q}(1)$
- simple objects: Tate objects $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ for $n \in \mathbb{Z}$
- $\mathrm{Ext}^1_{\mathsf{MT}(\mathbb{Z}_S,\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ for $n \leq 0$

There are realisation functors

$$\begin{aligned}
\mathrm{real}_{\text{ét},p} &\colon \mathsf{MT}(\mathbb{Z}_S, \mathbb{Q}) \to \mathsf{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}}),\\
\mathrm{real}_{\mathrm{dR}} &\colon \mathsf{MT}(\mathbb{Z}_S, \mathbb{Q}) \to \mathbb{Q}\text{-}\mathrm{Vect}_{\mathsf{f}},\\
\mathrm{real}_{\mathsf{cris},p} &\colon \mathsf{MT}(\mathbb{Z}_S, \mathbb{Q}) \to \mathrm{MF}^{\varphi,\mathrm{adm}}_{\mathbb{Q}_p} \quad (p \notin S).
\end{aligned}$$

[5] *Groupes fondamentaux motiviques de Tate mixte*, 2005

Motivic fundamental group

Let $b = \vec{1}_0$ be the tangential base point of $X = \mathbb{P}^1_{\mathbb{Z}_S} \smallsetminus \{0, 1, \infty\}$ based at 0.

The unipotent motivic fundamental group of X is a pro-unipotent group

$$\pi_1^{\mathrm{mot}}(X, b)$$

in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$.

Its realisations are the p-adic étale / de Rham / crystalline fundamental group:

$$\begin{aligned} \mathrm{real}_{\text{ét},p}(\pi_1^{\mathrm{mot}}(X, b)) &= \pi_1^{\text{ét},\mathbb{Q}_p}(X_{\overline{\mathbb{Q}}}, b), \\ \mathrm{real}_{\mathrm{dR}}(\pi_1^{\mathrm{mot}}(X, b)) &= \pi_1^{\mathrm{dR}}(X, b), \\ \mathrm{real}_{\mathrm{cris},p}(\pi_1^{\mathrm{mot}}(X, b)) &= \pi_1^{\mathrm{cris}}(X_{\mathbb{F}_p}, b). \end{aligned}$$

Let

$$\pi_1^{\mathrm{mot}}(X, b) \twoheadrightarrow \Pi$$

be a quotient of the motivic fundamental group in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$.

The motivic Selmer scheme is defined as the moduli space of Π-torsors in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$:

Definition/Theorem

The motivic Selmer scheme $\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X)$ *is the* $\mathbb{Q}$*-scheme representing the functor on* $\mathbb{Q}$*-algebras*

$$R \mapsto \{\Pi\text{-}\textit{torsors over } R \textit{ in } \mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})\}/\textit{iso}.$$

The motivic Kummer map

For $x \in X(\mathbb{Z}_S)$, we have the motivic path space $\pi_1^{\mathrm{mot}}(X; b, x)$ which is a $\pi_1^{\mathrm{mot}}(X, b)$-torsor in $\mathrm{MT}(\mathbb{Z}_S, \mathbb{Q})$. Let

$$\pi_1^{\mathrm{mot}}(X; b, x) \twoheadrightarrow {}_x\Pi_b$$

be the pushout along $\pi_1^{\mathrm{mot}}(X, b) \twoheadrightarrow \Pi$. This defines the motivic Kummer map

$$\begin{aligned} j_S \colon X(\mathbb{Z}_S) &\to \mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X)(\mathbb{Q}), \\ x &\mapsto [{}_x\Pi_b]. \end{aligned}$$

Algebraic group cohomology

Let k be a field, G/k a pro-algebraic group acting on a pro-unipotent group Π/k.

Definition

Let R be a k-algebra. An algebraic cocycle $c\colon G_R \to \Pi_R$ is a morphism of R-schemes satisfying the cocycle condition

$$c(gh) = c(g) \cdot g(c(h))$$

for all $g, h \in G_R$. Two cocycles c_1, c_2 are cohomologous if there exists $\gamma \in \Pi(R)$ such that

$$c_2(g) = \gamma^{-1} \cdot c_1(g) \cdot g(c(\gamma))$$

for all $g \in G_R$.

Algebraic group cohomology

Let

$$\mathsf{H}^1(G_R, \Pi_R) := \mathsf{Z}^1(G_R, \Pi_R)/\Pi(R)$$

be the pointed set of cohomology classes of algebraic cocycles. Write

$$\mathsf{H}^1(G, \Pi)$$

for the associated functor on R-algebras, $R \mapsto \mathsf{H}^1(G_R, \Pi_R)$.

Parametrising torsors in Tannakian categories

Let $\mathcal{T} = (\mathcal{T}, \otimes, 1)$ a k-linear Tannakian category and Π a pro-unipotent group in $\mathcal{T}$. Let $\omega\colon \mathcal{T} \to k\text{-}\mathrm{Vect}_{\mathrm{f}}$ be a fibre functor and $\pi_1(\mathcal{T}, \omega) := \mathrm{Aut}^{\otimes}(\omega)$ the associated Tannaka group.

Proposition

There is a canonical isomorphism of functors of pointed sets on k-algebras

$$\{\Pi\text{-}\textit{torsors in } \mathcal{T}\}/\textit{iso} \cong \mathrm{H}^1(\pi_1(\mathcal{T}, \omega), \omega(\Pi)).$$

Proof.

Let P be a Π-torsor over R in $\mathcal{T}$.

- $\omega(P)$ is a $\pi_1(\mathcal{T}, \omega)$-equivariant $\omega(\Pi)$-torsor over R.
- Π pro-unipotent $\Rightarrow \exists\ p \in \omega(P)(R)$
- $c_P\colon \pi_1(\mathcal{T}, \omega)_R \to \omega(\Pi)_R$, $g \mapsto p^{-1} \cdot g(p)$ is algebraic cocycle
- $P \mapsto [c_P]$ is well-defined bijection □

Let

$$G_S := \pi_1(\mathsf{MT}(\mathbb{Z}_S, \mathbb{Q}), \mathrm{real}_{\mathrm{dR}})$$

be the Tannaka group of $\mathsf{MT}(\mathbb{Z}_S, \mathbb{Q})$ with respect to the de Rham fibre functor. It is called the mixed Tate motivic Galois group of $\mathbb{Z}_S$.

Let $\pi_1^{\mathrm{mot}}(X, b) \twoheadrightarrow \Pi$ be a quotient in $\mathsf{MT}(\mathbb{Z}_S, \mathbb{Q})$.

Its de Rham realisation is a G_S-equivariant quotient

$$\pi_1^{\mathrm{dR}}(X, b) \twoheadrightarrow \Pi^{\mathrm{dR}}.$$

Corollary

The motivic Selmer scheme is isomorphic to the algebraic group cohomology

$$\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X) \cong \mathrm{H}^1(G_S, \Pi^{\mathrm{dR}}).$$

The mixed Tate motivic Galois group

The action of G_S on $\mathbb{Q}(-1)$ defines a surjective homomorphism $G_S \twoheadrightarrow \mathbb{G}_m$. The kernel is a pro-unipotent group U_S:

$$1 \to U_S \to G_S \to \mathbb{G}_m \to 1. \qquad (*)$$

Objects M of $\mathsf{MT}(\mathbb{Z}_S, \mathbb{Q})$ carry a natural weight filtration $W_\bullet M$ indexed by even integers s.t. $\operatorname{gr}^W_{2n} M$ is a direct sum of copies of $\mathbb{Q}(-n)$.

The splitting of the weight filtration on $\mathrm{real}_{\mathrm{dR}}(M)$ defines a splitting of $(*)$, so G_S is a semi-direct product

$$G_S = U_S \rtimes \mathbb{G}_m.$$

$\mathbb{G}_m$-invariant points of Π-torsors

Lemma

Let P be a G_S-equivariant Π^{dR}-torsor over a $\mathbb{Q}$-algebra R. Then $P(R)$ contains a unique $\mathbb{G}_m$-invariant point.

Proof.

$R = \mathbb{Q}$ for simplicity.

- Uniqueness: equivalent to $\Pi^{\mathrm{dR}}(\mathbb{Q})^{\mathbb{G}_m} = \{1\}$. Follows from $\mathrm{Lie}(\pi_1^{\mathrm{dR}}(X, b))$ being graded in strictly negative weights.
- Existence: equivalent to $\mathrm{H}^1(\mathbb{G}_m, \Pi^{\mathrm{dR}}) = \{*\}$. In general, $\mathrm{H}^1(G, U) = \{*\}$ for G reductive, U pro-unipotent. Proof for $U = V$ vector group: $\mathrm{H}^1(G, V)$ classifies extensions of G-representations

$$0 \to V \to E \to \mathbb{Q} \to 0.$$

G reductive $\Rightarrow$ every extension splits $\square$

Motivic Selmer scheme via $\mathbb{G}_m$-equivariant cocycles

Let P be a G_S-equivariant Π^{dR}-torsor over a $\mathbb{Q}$-algebra R.

Let p^{dR} be the unique $\mathbb{G}_m$-invariant point in $P(R)$.

It defines a canonical cocycle

$$c_P^{\mathsf{can}} \colon (G_S)_R \to \Pi_R^{\mathrm{dR}}, \quad g \mapsto (p^{\mathrm{dR}})^{-1} \cdot g(p^{\mathrm{dR}}).$$

Its restriction to U_S is $\mathbb{G}_m$-equivariant.

Proposition

The construction $P \mapsto c_P^{\mathsf{can}}|_{U_S}$ defines an isomorphism

$$\mathrm{H}^1(G_S, \Pi^{\mathrm{dR}}) \cong \mathrm{Z}^1(U_S, \Pi^{\mathrm{dR}})^{\mathbb{G}_m}.$$

Here, $\mathrm{Z}^1(U_S, \Pi^{\mathrm{dR}})^{\mathbb{G}_m}$ denotes the $\mathbb{G}_m$-equivariant cocycles $U_S \to \Pi^{\mathrm{dR}}$ (as a functor on $\mathbb{Q}$-algebras).

Nonabelian Lie algebra cocycles

Let $\mathfrak{u}$ and $\mathfrak{p}$ be Lie algebras over a field k.

Assume $\mathfrak{u}$ acts on $\mathfrak{p}$ by derivations via

$$\phi\colon \mathfrak{u} \to \mathrm{Der}(\mathfrak{p}), \quad X \mapsto \phi_X.$$

Definition

A 1-cocycle $C : \mathfrak{u} \to \mathfrak{p}$ is a linear map satisfying

$$C[X_1, X_2] = [CX_1, CX_2] + \phi_{X_1}(CX_2) - \phi_{X_2}(CX_1) \quad \text{for } X_1, X_2 \in \mathfrak{u}.$$

The vector space of cocycles is denoted by

$$\mathrm{Z}^1(\mathfrak{u}, \mathfrak{p}).$$

Nonabelian Lie algebra cocycles

Assume U and Π are algebraic groups with U acting on Π.
Then $\mathfrak{u} := \mathrm{Lie}(U)$ acts on $\mathfrak{p} := \mathrm{Lie}(\Pi)$ by derivations.

If $c\colon U \to \Pi$ is an algebraic group cocycle, the derivative at the identity is a Lie algebra cocycle. This defines a map

$$\mathrm{Z}^1(U, \Pi) \to \mathrm{Z}^1(\mathfrak{u}, \mathfrak{p}).$$

Lemma

If U and Π are unipotent, this is an isomorphism.

Proof.

$$\begin{array}{ccc}
\mathrm{Z}^1(U,\Pi) & \xrightarrow{\cong} & \{\text{homomorphic sections of } \Pi \rtimes U \to U\} \\
\downarrow & & \downarrow\cong \\
\mathrm{Z}^1(\mathfrak{u},\mathfrak{p}) & \xrightarrow{\cong} & \{\text{homomorphic sections of } \mathfrak{p} \rtimes \mathfrak{u} \to \mathfrak{u}\}
\end{array}$$

□

Motivic Selmer scheme via Lie algebra cocycles

Let $\pi_1^{\mathsf{mot}}(X, b) \twoheadrightarrow \Pi$ be a quotient in $\mathsf{MT}(\mathbb{Z}_S, \mathbb{Q})$.

We have isomorphisms

$$\begin{aligned} \mathsf{Sel}^{\mathsf{mot}}_{S,\Pi}(X) &\cong \mathsf{H}^1(G_S, \Pi^{\mathrm{dR}}) \\ &\cong \mathsf{Z}^1(U_S, \Pi^{\mathrm{dR}})^{\mathbb{G}_m} \\ &\cong \mathsf{Z}^1(\mathrm{Lie}(U_S), \mathrm{Lie}(\Pi^{\mathrm{dR}}))^{\mathbb{G}_m}. \end{aligned}$$

In other words, points of the motivic Selmer scheme are in bijection with graded cocycles of Lie algebras $\mathrm{Lie}(U_S) \to \mathrm{Lie}(\Pi^{\mathrm{dR}})$.

Structure of $\mathrm{Lie}(U_S)$

Proposition

$\mathrm{Lie}(U_S)$ is a free graded pro-nilpotent Lie algebra on generators

$$\Sigma = \{\tau_\ell : \ell \in S\} \cup \{\sigma_3, \sigma_5, \sigma_7, \ldots\}$$

with half-weights $\deg(\tau_\ell) = -1$, $\deg(\sigma_{2i+1}) = -(2i+1)$.

Proof.

Generators from

$$\mathrm{Ext}^1_{\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(\mathbb{Z}_S)_\mathbb{Q} = \begin{cases} \mathbb{Z}_S^\times \otimes \mathbb{Q}, & n = 1, \\ \dim 1, & n \geq 3 \text{ odd}, \\ 0, & o/w, \end{cases}$$

relations from $\mathrm{Ext}^2_{\mathrm{MT}(\mathbb{Z}_S,\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$. □

Motivic Selmer scheme as an affine space

Let $\Sigma = \coprod_{n<0} \Sigma_n = \{\tau_\ell : \ell \in S\} \cup \{\sigma_3, \sigma_5, \ldots\}$ be a choice of free generators of $\mathrm{Lie}(U_S)$ with Σ_n in degree n.

Let $\mathrm{Lie}(\Pi^{\mathrm{dR}}) = \prod_{n<0} \mathrm{Lie}(\Pi^{\mathrm{dR}})_n$ be the grading by half-weight.

Proposition

We have an isomorphism

$$\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X) \cong \prod_{n<0} \mathrm{Lie}(\Pi^{\mathrm{dR}})_n^{\Sigma_n}.$$

Proof.

We showed $\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X) = \mathrm{Z}^1(\mathrm{Lie}(U_S), \mathrm{Lie}(\Pi)^{\mathrm{dR}})^{\mathbb{G}_m}$.
Cocycles $C\colon \mathrm{Lie}(U_S) \to \mathrm{Lie}(\Pi^{\mathrm{dR}})$ are uniquely determined by the images of free generators. Graded cocycles map Σ_n into $\mathrm{Lie}(\Pi^{\mathrm{dR}})_n$. □

Motivic Selmer scheme as an affine space

Corollary

If Π is finite-dimensional, then $\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X)$ is an affine space over $\mathbb{Q}$ of dimension

$$\#S \cdot \dim \mathrm{Lie}(\Pi^{\mathrm{dR}})_{-1} + \sum_{i \geq 1} \dim \mathrm{Lie}(\Pi^{\mathrm{dR}})_{-(2i+1)}.$$

Proof.

$$\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X) \cong \prod_{n<0} \mathrm{Lie}(\Pi^{\mathrm{dR}})_n^{\Sigma_n}$$

Free generators of $\mathrm{Lie}(U_S)$ are

$$\begin{aligned} \Sigma_{-1} &= \{\tau_\ell : \ell \in S\}, \\ \Sigma_{-(2i+1)} &= \{\sigma_{2i+1}\}, \\ \Sigma_n &= \emptyset \quad \text{otherwise.} \end{aligned}$$

□

3. Coordinates on the Selmer scheme

The cocommutative Hopf algebra of $\pi_1^{\mathrm{dR}}(X, b)$

Assume $\Pi = \pi_1^{\mathrm{mot}}(X, b)$ is the full fundamental group.

Then $\mathrm{Lie}(\Pi^{\mathrm{dR}})$ is a free pro-nilpotent Lie algebra on two generators e_0 and e_1 in degree -1.

Let

$$\mathcal{H}(\Pi^{\mathrm{dR}}) := \mathcal{O}(\Pi^{\mathrm{dR}})^{\vee} \cong \mathcal{U}(\mathrm{Lie}(\Pi^{\mathrm{dR}}))$$

be the co-commutative Hopf algebra associated to Π^{dR}. Then

$$\mathcal{H}(\Pi^{\mathrm{dR}}) = \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$$

is a non-commutative power series algebra on e_0, e_1. Elements are $\sum_w a_w \cdot w$, $a_w \in \mathbb{Q}$, with w running through words in e_0, e_1.

Lie-like elements

$$\mathcal{H}(\Pi^{\mathrm{dR}}) = \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$$

counit: $\varepsilon(\sum_w a_w\, w) = a_\emptyset$
coproduct: $\Delta e_i = e_i \otimes 1 + 1 \otimes e_i$

Definition

An element $A \in \mathbb{Q}\langle\langle e_0, e_1 \rangle\rangle$ is Lie-like (or primitive) if

$$\varepsilon(A) = 0 \quad \text{and} \quad \Delta(A) = A \otimes 1 + 1 \otimes A.$$

Remark

$\sum a_w\, w$ is Lie-like iff $a_\emptyset = 0$ and for all non-empty w_1, w_2, we have

$$\sum_{\sigma \in \mathrm{Sh}(|w_1|,|w_2|)} a_{\sigma(w_1 w_2)} = 0,$$

where $\mathrm{Sh}(m, n) \subseteq S_{m+n}$ denotes the shuffle permutations.

The Lie algebra $\mathrm{Lie}(\Pi^{\mathrm{dR}})$ embeds into $\mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ as the Lie-like elements with the Lie bracket $[A, B] = AB - BA$.

Given a word w in e_0, e_1, let

$$f_w : \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle \to \mathbb{Q}$$

be the coefficient extraction functional $\sum_{w'} a_{w'} w' \mapsto a_w$.

If w has degree n, the restriction of f_w to $\mathrm{Lie}(\Pi^{\mathrm{dR}})$ defines an element of $\mathrm{Lie}(\Pi^{\mathrm{dR}})_n^\vee$.

Lyndon basis

Definition

A word in the symbols e_0, e_1 is a Lyndon word if it is non-empty and lexicographically smaller (with respect to the ordering $e_0 < e_1$) than its cyclic rotations.

Examples:

$$e_0, \quad e_1, \quad e_0 e_1, \quad e_0 e_0 e_1, \quad e_0 e_1 e_1, \quad e_0 e_0 e_0 e_1, \quad e_0 e_0 e_1 e_1, \quad e_0 e_1 e_1 e_1.$$

Proposition

The coefficient extraction functionals f_w with w running through the Lyndon words of degree n form a basis of $\mathrm{Lie}(\Pi^{\mathrm{dR}})_n^\vee$.

Coordinates on the Selmer scheme

Let $\Sigma = \{\tau_\ell : \ell \in S\} \cup \{\sigma_3, \sigma_5, \ldots\}$ be free generators of $\mathrm{Lie}(U_S)$. For a graded cocycle $c\colon \mathrm{Lie}(U_S) \to \mathrm{Lie}(\Pi^{\mathrm{dR}})$ define

$$x_\ell(c) := f_{e_0}(c(\tau_\ell)),$$
$$y_\ell(c) := f_{e_1}(c(\tau_\ell)),$$

for $\ell \in S$, and

$$z_w(c) := f_w(c(\sigma_{2i+1}))$$

for $i \geq 1$ and Lyndon words of length $2i + 1$.

Theorem

$\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X)$ *is an affine space with coordinates* x_ℓ, y_ℓ, z_w *as above.*

Proof.

Graded cocycles are uniquely determined by $c(\tau_\ell) \in \mathrm{Lie}(\Pi^{\mathrm{dR}})_{-1}$ and $c(\sigma_{2i+1}) \in \mathrm{Lie}(\Pi^{\mathrm{dR}})_{-(2i+1)}$. □

Have similar results for quotients of $\pi_1^{\mathrm{mot}}(X, b)$.

Example

Π_{PL} = polylogarithmic quotient.

$\mathrm{Lie}(\Pi_{\mathrm{PL}}^{\mathrm{dR}})_{-1}^{\vee}$ has basis $\{f_{e_0}, f_{e_1}\}$.
$\mathrm{Lie}(\Pi_{\mathrm{PL}}^{\mathrm{dR}})_{-(2i+1)}^{\vee}$ has basis $\{f_{e_0^{2i}e_1}\}$.

$$\mathrm{Sel}_{S,\mathrm{PL}}^{\mathrm{mot}}(X) = \mathrm{Spec}\,\mathbb{Q}[(x_\ell)_{\ell\in S}, (y_\ell)_{\ell\in S}, z_3, z_5, \ldots],$$

where $z_{2i+1}(c) := z_{e_0^{2i}e_1}(c) = f_{e_0^{2i}e_1}(c(\sigma_{2i+1}))$.

Coordinates on the 4-step nilpotent Selmer scheme

Example

Π_4 = 4-step nilpotent quotient.

$\mathrm{Lie}(\Pi_4^{\mathrm{dR}})^\vee_{-3}$ has basis $\{f_{e_0e_0e_1}, f_{e_0e_1e_1}\}$ corresponding to the two Lyndon words of length 3.

$$\mathrm{Sel}^{\mathrm{mot}}_{S,4}(X) = \mathrm{Spec}\, \mathbb{Q}[(x_\ell)_{\ell\in S}, (y_\ell)_{\ell\in S}, z_3, z_{2,1}],$$

where

$$z_3(c) := z_{e_0e_0e_1}(c) = f_{e_0e_0e_1}(c(\sigma_3)),$$
$$z_{2,1}(c) := z_{e_0e_1e_1}(c) = f_{e_0e_1e_1}(c(\sigma_3))$$

4. Chabauty–Kim calculations

Motivic Chabauty–Kim diagram:

$$
\begin{array}{ccc}
X(\mathbb{Z}_S) & \hookrightarrow & X(\mathbb{Z}_p) \\
\downarrow j_S & & \downarrow j_p \\
\mathrm{Z}^1(U_S, \Pi^{\mathrm{dR}})^{\mathbb{G}_m}(\mathbb{Q}) & \xrightarrow{\mathrm{loc}_p} & \Pi^{\mathrm{dR}}(\mathbb{Q}_p)
\end{array}
$$

The localisation map is given by evaluating cocycles at Chatzistamatiou–Ünver's *p*-adic period point $\eta_p \in U_S(\mathbb{Q}_p)$:

$$\mathrm{loc}_p(c) = c(\eta_p).$$

Remark: We show in [BKL23][6] that this motivic Chabauty–Kim diagram is isomorphic to the classical one defined via Galois cohomology of the $\mathbb{Q}_p$-prounipotent étale fundamental group.

[6] *Chabauty–Kim and the Section Conjecture for locally geometric sections*

Grouplike elements

Let

$$\mathcal{H}(U_S) := \mathcal{O}(U_S)^\vee \cong \mathcal{U}(\mathrm{Lie}(U_S))$$

be the co-commutative Hopf algebra of U_S.

Let $\Sigma = \{\tau_\ell : \ell \in S\} \cup \{\sigma_3, \sigma_5, \ldots\}$ be free generators of $\mathrm{Lie}(U_S)$.
Then

$$\mathcal{H}(U_S) = \mathbb{Q}\langle\langle\Sigma\rangle\rangle$$

is the non-commutative power series algebra on Σ.

Definition

Let R be a $\mathbb{Q}$-algebra. An element $g \in R\langle\langle\Sigma\rangle\rangle$ is grouplike if

$$\varepsilon(g) = 1 \quad \text{and} \quad \Delta(g) = g \otimes g.$$

Remark

$g = \sum a_u\, u$ is grouplike iff $a_\emptyset = 1$ and for all u_1, u_2 we have the shuffle relation $a_{u_1} a_{u_2} = \sum_{\sigma \in \mathrm{Sh}(|u_1|,|u_2|)} a_{\sigma(u_1 u_2)}$.

Hopf algebra cocycles

$U_S(R)$ embeds into $R\langle\langle\Sigma\rangle\rangle^\times$ as the grouplike elements.

The group action of U_S on Π^{dR} extends to a Hopf algebra action of $\mathcal{H}(U_S) = \mathbb{Q}\langle\langle\Sigma\rangle\rangle$ on $\mathcal{H}(\Pi^{\mathrm{dR}}) = \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$.

Any group cocycle $c\colon U_S \to \Pi^{\mathrm{dR}}$ extends to a Hopf algebra cocycle

$$c\colon \mathbb{Q}\langle\langle\Sigma\rangle\rangle \to \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle.$$

Cocycle condition (using Sweedler's notation):

$$c(XY) = \sum_{(X)} c(X_{(1)}) \cdot X_{(2)}.c(Y).$$

In particular, if X is Lie-like:

$$c(XY) = c(X)c(Y) + X.c(Y).$$

Three kinds of cocycles

Cocycles between algebraic groups, their Lie algebras and their Hopf algebras are equivalent:

$$\mathsf{Z}^1(U_S, \Pi^{\mathrm{dR}}) \cong \mathsf{Z}^1(\mathrm{Lie}(U_S), \mathrm{Lie}(\Pi^{\mathrm{dR}})) \cong \mathsf{Z}^1(\mathcal{H}(U_S), \mathcal{H}(\Pi^{\mathrm{dR}}))$$

For explicit calculations, working with the Hopf algebras $\mathcal{H}(U_S) = \mathbb{Q}\langle\langle\Sigma\rangle\rangle$ and $\mathcal{H}(\Pi^{\mathrm{dR}}) = \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle$ is most convenient.

Calculating the localisation map

We want to calculate the cocycle evaluation map

$$\mathrm{loc}_p \colon \mathrm{Z}^1(U_S, \Pi^{\mathrm{dR}})^{\mathbb{G}_m}_{\mathbb{Q}_p} \to \Pi^{\mathrm{dR}}_{\mathbb{Q}_p},$$
$$c \mapsto c(\eta_p).$$

Write the period point $\eta_p \in U_S(\mathbb{Q}_p)$ as a grouplike power series:

$$\eta_p = \sum_u a_u \, u \in \mathbb{Q}_p\langle\langle \Sigma \rangle\rangle.$$

Functions on Π^{dR} are $\mathrm{Li}_w := f_w$ for words w in e_0, e_1.

Let $c \colon \mathbb{Q}_p\langle\langle \Sigma \rangle\rangle \to \mathbb{Q}_p\langle\langle e_0, e_1 \rangle\rangle$ be a graded Hopf algebra cocycle.

$$(\mathrm{loc}_p^\sharp \, \mathrm{Li}_w)(c) = f_w(c(\eta_p)) = \sum_u a_u f_w(c(u))$$

Localisation map in depth 1

$$(\mathrm{loc}_p^\sharp \mathrm{Li}_w)(c) = \sum_u a_u f_w(c(u))$$

Since c is graded, we can restrict the sum to those u of the same degree as w.

Example: Words over Σ of degree -1 are τ_ℓ for $\ell \in S$.

$$(\mathrm{loc}_p^\sharp \log)(c) = (\mathrm{loc}_p^\sharp \mathrm{Li}_{e_0})(c) = \sum_{\ell \in S} a_{\tau_\ell} f_{e_0}(c(\tau_\ell)) = \sum_{\ell \in S} a_{\tau_\ell} x_\ell(c),$$
$$(\mathrm{loc}_p^\sharp \mathrm{Li}_1)(c) = (\mathrm{loc}_p^\sharp \mathrm{Li}_{e_1})(c) = \sum_{\ell \in S} a_{\tau_\ell} f_{e_1}(c(\tau_\ell)) = \sum_{\ell \in S} a_{\tau_\ell} y_\ell(c).$$

Localisation map in depth 2

Words over Σ of degree -2 are $\tau_\ell \tau_q$ with $\ell, q \in S$.
To evaluate $c(\tau_\ell \tau_q)$ use:

- cocycle property: $c(\tau_\ell \tau_q) = c(\tau_\ell)c(\tau_q) + \tau_\ell.c(\tau_q)$
- τ_ℓ acts trivially on $\mathrm{Lie}(\Pi^{\mathrm{dR}})$, so $\tau_\ell.c(\tau_q) = 0$

$$\Rightarrow \; c(\tau_\ell \tau_q) = c(\tau_\ell)c(\tau_q)$$

$$\begin{aligned}
\Rightarrow (\mathrm{loc}_p^\sharp \mathrm{Li}_2)(c) = (\mathrm{loc}_p^\sharp \mathrm{Li}_{e_0 e_1})(c) &= \sum_{\ell, q \in S} a_{\tau_\ell \tau_q} f_{e_0 e_1}(c(\tau_\ell \tau_q)) \\
&= \sum_{\ell, q \in S} a_{\tau_\ell \tau_q} f_{e_0 e_1}(c(\tau_\ell) c(\tau_q)) \\
&= \sum_{\ell, q \in S} a_{\tau_\ell \tau_q} f_{e_0}(c(\tau_\ell)) f_{e_1}(c(\tau_q)) \\
&= \sum_{\ell, q \in S} a_{\tau_\ell \tau_q} x_\ell(c) y_q(c)
\end{aligned}$$

Localisation map in depth 2

We recover the localisation map for the 2-step nilpotent quotient Π_2 as previously derived by Dan-Cohen and Wewers:

$$\mathrm{loc}_p \colon \operatorname{Spec} \mathbb{Q}_p[(x_\ell)_{\ell \in S}, (y_\ell)_{\ell \in S}] \to \operatorname{Spec} \mathbb{Q}_p[\log, \mathrm{Li}_1, \mathrm{Li}_2]$$

$$\begin{aligned} \mathrm{loc}_p^\sharp \log &= \sum_{\ell \in S} a_{\tau_\ell} x_\ell, \\ \mathrm{loc}_p^\sharp \mathrm{Li}_1 &= \sum_{\ell \in S} a_{\tau_\ell} y_\ell, \\ \mathrm{loc}_p^\sharp \mathrm{Li}_2 &= \sum_{\ell, q \in S} a_{\tau_\ell \tau_q} x_\ell y_q. \end{aligned}$$

In higher depth we have to evaluate things like

$$c(\sigma_3 \tau_\ell) = c(\sigma_3) c(\tau_\ell) + \sigma_3 . c(\tau_\ell),$$

so we need to understand the motivic Galois action.

Let

$$\kappa_1 \colon \mathrm{Lie}(U_S) \to \mathrm{Lie}(\Pi^{\mathrm{dR}})$$

be the cocycle representing the path torsor $\pi_1^{\mathsf{mot}}(X; \vec{1}_0, -\vec{1}_1)$.
It is given by

$$\kappa_1(\sigma) = (p^{\mathrm{dR}})^{-1}\sigma(p^{\mathrm{dR}})$$

where p^{dR} is the unique $\mathbb{G}_m$-invariant path from $\vec{1}_0$ to $-\vec{1}_1$.

Theorem

The action of $\mathrm{Lie}(U_S)$ on $\mathrm{Lie}(\Pi^{\mathrm{dR}})$ satisfies and is completely determined by the following:

1. *the τ_ℓ act trivially;*
2. $\sigma.e_0 = 0$ *for all* $\sigma \in \mathrm{Lie}(U_S)$*;*
3. $\sigma.e_1 = [e_1, \kappa_1(\sigma)]$ *for all* $\sigma \in \mathrm{Lie}(U_S)$*.*

Motivic multiple zeta values

For a tuple $(k_1, \ldots, k_r)$ define the motivic multiple zeta value

$$\zeta^{\mathfrak{u}}(k_1, \ldots, k_r) \in \mathcal{O}(U_S)$$

as the function on U_S given by

$$u \mapsto \int_{u(p^{\mathrm{dR}})} \underbrace{\frac{\mathrm{d}z}{z} \cdots \frac{\mathrm{d}z}{z} \frac{\mathrm{d}z}{1-z}}_{k_1} \cdots \underbrace{\frac{\mathrm{d}z}{z} \cdots \frac{\mathrm{d}z}{z} \frac{\mathrm{d}z}{1-z}}_{k_r}.$$

Alternative notation: $\zeta^{\mathfrak{u}}(e_0^{k_1-1} e_1 \cdots e_0^{k_r-1} e_1)$.

With respect to the pairing

$$\langle \cdot, \cdot \rangle \colon \mathcal{O}(U_S) \otimes \mathcal{H}(U_S) \to \mathbb{Q}$$

we have for $\sigma \in \mathrm{Lie}(U_S)$ and words w in e_0, e_1:

$$f_w(\kappa_1(\sigma)) = \langle \zeta^{\mathfrak{u}}(w), \sigma \rangle.$$

A nontrivial calculation

In order to calculate $f_{e_0 e_1 e_1 e_1}(\sigma_3.c(\tau_\ell))$, combine:

- $c(\tau_\ell) = x_\ell(c)e_0 + y_\ell(c)e_1$
- σ_3 acts trivially on e_0
- $\sigma_3.e_1 = [e_1, \kappa_1(\sigma_3)]$
- $f_w(\kappa_1(\sigma_3)) = \langle \zeta^{\mathfrak{u}}(w), \sigma_3 \rangle$
- identity of motivic MZV: $\zeta^{\mathfrak{u}}(2,1) = \zeta^{\mathfrak{u}}(3)$
- $\langle \zeta^{\mathfrak{u}}(3), \sigma_3 \rangle = 1$ (for suitable choice of σ_3)

to obtain

$$f_{e_0 e_1 e_1 e_1}(\sigma_3.c(\tau_\ell)) = -y_\ell(c).$$

Localisation map in depth 4

$\mathrm{loc}_p \colon \operatorname{Spec} \mathbb{Q}_p[(x_\ell)_{\ell \in S}(y_\ell)_{\ell \in S}, z_3, z_{2,1}] \to \operatorname{Spec} \mathbb{Q}_p[\log, \mathrm{Li}_1, \mathrm{Li}_2, \mathrm{Li}_3, \mathrm{Li}_{2,1}, \mathrm{Li}_4, \mathrm{Li}_{3,1}, \mathrm{Li}_{2,1,1}]$

is given by

$$\mathrm{loc}_p^\sharp \log = \sum_{\ell \in S} a_{\tau_\ell} x_\ell,$$

$$\mathrm{loc}_p^\sharp \mathrm{Li}_1 = \sum_{\ell \in S} a_{\tau_\ell} y_\ell,$$

$$\mathrm{loc}_p^\sharp \mathrm{Li}_2 = \sum_{\ell, q \in S} a_{\tau_\ell \tau_q} x_\ell y_q,$$

$$\mathrm{loc}_p^\sharp \mathrm{Li}_3 = \sum_{\ell_1, \ell_2, q \in S} a_{\tau_{\ell_1} \tau_{\ell_2} \tau_q} x_{\ell_1} x_{\ell_2} y_q + a_{\sigma_3} z_3,$$

$$\mathrm{loc}_p^\sharp \mathrm{Li}_{2,1} = \sum_{\ell, q_1, q_2 \in S} a_{\tau_\ell \tau_{q_1} \tau_{q_2}} x_\ell y_{q_1} y_{q_2} + a_{\sigma_3} z_{2,1},$$

$$\mathrm{loc}_p^\sharp \mathrm{Li}_4 = \sum_{\ell_1, \ell_2, \ell_3, q \in S} a_{\tau_{\ell_1} \tau_{\ell_2} \tau_{\ell_3} \tau_q} x_{\ell_1} x_{\ell_2} x_{\ell_3} y_q + \sum_{\ell \in S} a_{\tau_\ell \sigma_3} x_\ell z_3,$$

$$\mathrm{loc}_p^\sharp \mathrm{Li}_{3,1} = \sum_{\ell_1, \ell_2, q_1, q_2 \in S} a_{\tau_{\ell_1} \tau_{\ell_2} \tau_{q_1} \tau_{q_2}} x_{\ell_1} x_{\ell_2} y_{q_1} y_{q_2} + \sum_{\ell \in S} a_{\tau_\ell \sigma_3} x_\ell z_{2,1} + \sum_{\ell \in S} a_{\sigma_3 \tau_\ell} y_\ell (z_3 - 1),$$

$$\mathrm{loc}_p^\sharp \mathrm{Li}_{2,1,1} = \sum_{\ell, q_1, q_2, q_3 \in S} a_{\tau_\ell \tau_{q_1} \tau_{q_2} \tau_{q_3}} x_\ell y_{q_1} y_{q_2} y_{q_3} + \sum_{\ell \in S} a_{\sigma_3 \tau_\ell} y_\ell (z_{2,1} - 1).$$

Refined equations for $S = \{2\}$

For $S = \{2\}$ and refinement (1), the localisation map simplifies to

$\mathrm{loc}_p \colon \operatorname{Spec} \mathbb{Q}_p[y, z_3, z_{2,1}] \to \operatorname{Spec} \mathbb{Q}_p[\log, \mathrm{Li}_1, \mathrm{Li}_2, \mathrm{Li}_3, \mathrm{Li}_{2,1}, \mathrm{Li}_4, \mathrm{Li}_{3,1}, \mathrm{Li}_{2,1,1}]$

given by

$$\begin{aligned}
\mathrm{loc}_p^\sharp \log &= 0,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_1 &= a_{\tau_2} y,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_2 &= 0,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_3 &= a_{\sigma_3} z_3,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_{2,1} &= a_{\sigma_3} z_{2,1},\\
\mathrm{loc}_p^\sharp \mathrm{Li}_4 &= 0,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_{3,1} &= a_{\sigma_3\tau_2} y(z_3 - 1),\\
\mathrm{loc}_p^\sharp \mathrm{Li}_{2,1,1} &= a_{\sigma_3\tau_2} y(z_{2,1} - 1).
\end{aligned}$$

So we find equations for $X(\mathbb{Z}_p)^{(1)}_{\{2\},\infty}$ like

$$a_{\sigma_3} a_{\tau_2} \mathrm{Li}_{2,1,1}(z) - a_{\sigma_3\tau_2} \mathrm{Li}_1(z)\big(\mathrm{Li}_{2,1}(z) - a_{\sigma_3}\big) = 0.$$

5. Summary

Summary

- Motivic Selmer scheme classifies $\pi_1^{\mathrm{mot}}(X, b)$-torsors
- More concrete descriptions via $\mathbb{G}_m$-equivariant cocycles:

$$\begin{aligned}\mathrm{Sel}^{\mathrm{mot}}_{S,\Pi}(X) &\cong \mathrm{Z}^1(U_S, \Pi^{\mathrm{dR}})^{\mathbb{G}_m}\\ &\cong \mathrm{Z}^1(\mathrm{Lie}(U_S), \mathrm{Lie}(\Pi^{\mathrm{dR}}))^{\mathbb{G}_m}\\ &\cong \{\text{graded Hopf cocycles } \mathbb{Q}\langle\langle\Sigma\rangle\rangle \to \mathbb{Q}\langle\langle e_0, e_1\rangle\rangle\}\end{aligned}$$

- Coordinates: $f_w(c(\sigma))$ for $\sigma \in \Sigma$, w Lyndon word in e_0, e_1
- Understand Galois action of $\mathrm{Lie}(U_S)$ on $\mathrm{Lie}(\Pi^{\mathrm{dR}})$ via motivic MZVs
- Can compute localisation map loc_p in coordinates
- Find new, non-polylogarithmic Coleman functions defining Chabauty–Kim loci