

Polylogarithmic Chabauty–Kim loci over number fields

Martin Lüdtkke

University of Oldenburg / BGU

Online Seminar
27 March 2026

joint with X. Li

The plan

Chabauty–Kim over number fields

Polylogarithmic Chabauty–Kim loci for the thrice-punctured line

The theory behind

1. Chabauty–Kim over number fields

S -integral points on curves

- ▶ K number field, S finite set of primes of K
- ▶ $\mathcal{O}_{K,S} = \mathcal{O}_K[S^{-1}]$ ring of S -integers
- ▶ $X/\mathcal{O}_{K,S}$ regular model of smooth hyperbolic curve

Theorem (Siegel, Mahler, Faltings)

$$\#X(\mathcal{O}_{K,S}) < \infty$$

Example

$$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad K = \mathbb{Q}(i), \quad S = \{(1-i)\}$$

$$\Rightarrow X(\mathbb{Z}[i, \frac{1}{1-i}]) = \left\{ 2, -1, \frac{1}{2}, i, -i, 1+i, 1-i, \frac{1+i}{2}, \frac{1-i}{2} \right\}$$

Chabauty–Kim over number fields

Fix

- ▶ a rational prime p which splits completely in K s.t. all $\mathfrak{p} \mid p$ are of good reduction and $\notin S$
- ▶ a G_K -equivariant quotient $\pi_1^{\text{ét}, \mathbb{Q}_p}(X_{\overline{K}}, b) \twoheadrightarrow \Pi^{\text{ét}}$

Aim: locate $X(\mathcal{O}_{K,S})$ inside $X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) \cong \prod_{\mathfrak{p} \mid p} X(\mathcal{O}_{\mathfrak{p}})$ or inside $X(\mathcal{O}_{\mathfrak{p}})$ for fixed $\mathfrak{p} \mid p$.

Chabauty–Kim method produces subsets (**Chabauty–Kim loci**)

$$\begin{aligned} X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{S, \Pi^{\text{ét}}} &\subseteq X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p), \\ X(\mathcal{O}_{\mathfrak{p}})_{S, \Pi^{\text{ét}}} &\subseteq X(\mathcal{O}_{\mathfrak{p}}) \end{aligned}$$

containing $X(\mathcal{O}_{K,S})$.

Remark

When X is projective and $\Pi^{\text{ét}} = \pi_1^{\text{ét}, \mathbb{Q}_p}(X_{\overline{K}}, b)^{\text{ab}}$, the CK method specialises to Siksek's **Restriction of Scalars Chabauty**.

Chabauty–Kim diagram

$$\begin{array}{ccc}
 X(\mathcal{O}_{K,S}) & \hookrightarrow & X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \prod_{p|p} X(\mathcal{O}_p) \\
 \downarrow j_S & & \downarrow j_p = \prod_p j_p \\
 \text{Sel}_{S, \Pi^{\text{ét}}}(X) & \xrightarrow{\text{loc}_p = \prod_p \text{loc}_p} & \prod_{p|p} H_f^1(G_p, \Pi^{\text{ét}})
 \end{array}$$

- ▶ $\text{Sel}_{S, \Pi^{\text{ét}}}(X) \subseteq H^1(G_K, \Pi^{\text{ét}})$ (**global Selmer scheme**) parametrises $\Pi^{\text{ét}}$ -torsors with G_K -action satisfying local conditions
- ▶ $H_f^1(G_p, \Pi^{\text{ét}})$ (**local Selmer scheme**) parametrises crystalline $\Pi^{\text{ét}}$ -torsors with G_p -action
- ▶ j_S and j_p (**global and local Kummer map**) form path torsors:

$$x \mapsto \pi_1^{\text{ét}, \mathbb{Q}_p}(X_{\overline{K}}; b, x) \times \pi_1^{\text{ét}, \mathbb{Q}_p}(X_{\overline{K}}, b) \times \Pi^{\text{ét}}$$

- ▶ loc_p restricts G_K -action to $G_p \subseteq G_K$

$$\begin{array}{ccc}
 X(\mathcal{O}_{K,S}) & \hookrightarrow & X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \prod_{p|p} X(\mathcal{O}_p) \\
 \downarrow j_S & & \downarrow j_p = \prod_p j_p \\
 \text{Sel}_{S, \Pi^{\text{ét}}}(X) & \xrightarrow{\text{loc}_p = \prod_p \text{loc}_p} & \prod_{p|p} H_f^1(G_p, \Pi^{\text{ét}})
 \end{array}$$

Fact: loc_p is a morphism of affine \mathbb{Q}_p -schemes

Definition (Chabauty–Kim loci)

$$X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{S, \Pi^{\text{ét}}} := j_p^{-1}(\text{loc}_p(\text{Sel}_{S, \Pi^{\text{ét}}}(X)))$$

$$X(\mathcal{O}_p)_{S, \Pi^{\text{ét}}} := \text{pr}_p(X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{S, \Pi^{\text{ét}}})$$

Here, $\text{pr}_p: X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \prod_{p'|p} X(\mathcal{O}_{p'}) \rightarrow X(\mathcal{O}_p)$ is the projection map.

Weaker variant for single prime $p \mid p$:

$$X(\mathcal{O}_p)'_{S, \Pi^{\text{ét}}} := j_p^{-1}(\text{loc}_p(\text{Sel}_{S, \Pi^{\text{ét}}}(X))) \supseteq X(\mathcal{O}_p)_{S, \Pi^{\text{ét}}}$$

Kim's Conjecture over number fields

Conjecture 1 (Kim's Conjecture)

For sufficiently large quotients $\pi_1^{\text{ét}, \mathbb{Q}_p}(X_{\overline{K}}, b) \twoheadrightarrow \Pi^{\text{ét}}$:

$$X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{S, \Pi^{\text{ét}}} = X(\mathcal{O}_{K, S}).$$

Conjecture 2 (Kim's Conjecture for $\mathfrak{p} \mid p$)

For sufficiently large quotients $\pi_1^{\text{ét}, \mathbb{Q}_p}(X_{\overline{K}}, b) \twoheadrightarrow \Pi^{\text{ét}}$:

$$X(\mathcal{O}_{\mathfrak{p}})_{S, \Pi^{\text{ét}}} = X(\mathcal{O}_{K, S}).$$

- ▶ Conjecture 1 \Rightarrow Conjecture 2 for all $\mathfrak{p} \mid p$
- ▶ converse can fail



Photo credit: Jan Vonk

Relevance of Kim's Conjecture

Why do we care about Kim's Conjecture?

- ▶ reduces determination of $X(\mathcal{O}_{K,S})$ to computing CK loci
- ▶ implies a weak form of Grothendieck's Section Conjecture:

S -Selmer Section Conjecture

Every locally geometric S -integral Galois section $s: G_K \rightarrow \pi_1^{\text{ét}}(X, b)$ comes from an S -integral point.

Theorem (Betts–Kumpitsch–L., 2023)

If $X/\mathcal{O}_{K,S}$ satisfies Conjecture 2 for a Dirichlet-dense set of primes \mathfrak{p} then it satisfies the S -Selmer Section Conjecture.

Dogra's finiteness results

- ▶ take $\Pi^{\text{ét}} = \Pi_n^{\text{ét}}$ the (nilpotency) depth n quotient
- ▶ locus $X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{S,n}$ is cut out in $X(\mathcal{O}_K \otimes \mathbb{Z}_p) = \prod_{p|p} X(\mathcal{O}_p)$ by iterated Coleman integrals of length $\leq n$
- ▶ at least $[K : \mathbb{Q}]$ functions needed to cut out a finite set
- ▶ need to rule out unlikely intersections

Theorem (Dogra, 2023)

If $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ then $\#X(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{S,n} < \infty$ for $n \gg 0$.

Theorem (Dogra, 2023)

Suppose X/K is smooth projective of genus > 1 . Assume that the conjecture of Jannsen, or the conjecture of Bloch–Kato, hold for X^n for all n . Then $\#X(K \otimes_{\mathbb{Q}} \mathbb{Q}_p)_n < \infty$ for $n \gg 0$.

2. Polylogarithmic Chabauty–Kim loci for the thrice-punctured line

Polylogarithmic Chabauty–Kim loci

From now on:

- ▶ $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$
- ▶ $\Pi^{\text{ét}} = \Pi_{\text{PL},n}^{\text{ét}}$ **depth- n polylogarithmic quotient** ($1 \leq n \leq \infty$)

\Rightarrow CK loci are cut out by polynomials in p -adic (poly)logarithms $\log, \text{Li}_1, \dots, \text{Li}_n: X(\mathcal{O}_p) \rightarrow K_p$

$$\log(z) = \int_{\bar{1}_0}^z \frac{dt}{t}, \quad \text{Li}_k(z) = \int_{\bar{1}_0}^z \underbrace{\frac{dt}{t} \cdots \frac{dt}{t} \frac{dt}{1-t}}_k$$

Notation for **polylogarithmic Chabauty–Kim loci**:

$$X(\mathcal{O}_{K,S} \otimes \mathbb{Z}_p)_{S,\text{PL},n} \quad \text{resp.} \quad X(\mathcal{O}_p)_{S,\text{PL},n} \quad \text{resp.} \quad X(\mathcal{O}_p)'_{S,\text{PL},n}$$

What is known

Theorem (Kim, 2012)

If K is totally real then $\#X(\mathcal{O}_{\mathfrak{p}})'_{S, \text{PL}, n}$ is finite for $n \gg 0$.

Theorem (Dan-Cohen, Wewers, 2015)

For K a real quadratic field, $S = \emptyset$, $n = 2$, we have

$$X(\mathcal{O}_{\mathfrak{p}})'_{\emptyset, \text{PL}, 2} = \left\{ z \in X(\mathbb{Z}_p) \mid \text{Li}_2(z) - \frac{1}{2} \log(z) \log(1-z) = 0 \right\}$$

Theorem (Dan-Cohen, 2020)

When K is not totally real then $X(\mathcal{O}_{\mathfrak{p}})'_{S, \text{PL}, \infty} = X(\mathcal{O}_{\mathfrak{p}})$.¹

¹except when K is imaginary quadratic and $S = \emptyset$

Theorem (Li-L.)

For K imaginary quadratic, $S = \emptyset$, any $1 \leq n \leq \infty$, the locus

$$X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{\emptyset, \text{PL}, n}$$

is contained in the set of pairs $(z_1, z_2) \in X(\mathbb{Z}_p) \times X(\mathbb{Z}_p)$ satisfying

$$\begin{aligned} \log(z_1) &= \log(z_2) = 0, \\ \log(1 - z_1) &= \log(1 - z_2) = 0, \\ \text{Li}_k(z_1) + (-1)^k \text{Li}_k(z_2) &= 0 \text{ for } 2 \leq k \leq n. \end{aligned}$$

The containment is an equality assuming a p -adic version of Grothendieck's period conjecture.

Primitive sixth root of unity

We can solve the equations using the following lemma:

Lemma

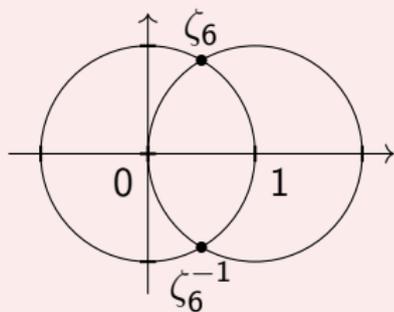
Let $z \in \mathbb{Z}_p$ with $z \not\equiv 0, 1 \pmod{p}$. If $\log(z) = 0$ and $\log(1 - z) = 0$, then z is a primitive 6-th root of unity.

Proof.

If $\log(z) = 0$ and $\log(1 - z) = 0$ then both z and $1 - z$ are roots of unity. Fix an embedding of $\mathbb{Q}(z)$ into \mathbb{C} . Then

$$|z|_{\mathbb{C}} = 1 \text{ and } |1 - z|_{\mathbb{C}} = 1.$$

The points of intersection of the unit circles centered at 0 and at 1 are $\zeta_6 = e^{2\pi i/6}$ and $\zeta_6^{-1} = e^{-2\pi i/6}$. \square



Kim's Conjecture for $\mathbb{Z}[\zeta_3]$ -points

The lemma implies:

$$\begin{aligned}X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{\emptyset,1} &= \mu_6^{\text{prim}}(\mathbb{Z}_p) \times \mu_6^{\text{prim}}(\mathbb{Z}_p) \\X(\mathcal{O}_p)_{\emptyset,1} &= \mu_6^{\text{prim}}(\mathbb{Z}_p)\end{aligned}$$

Note that $X(\mathbb{Z}[\zeta_3]) = \{\zeta_6, \zeta_6^{-1}\}$ since $\zeta_6 + \zeta_6^{-1} = 1$.

Corollary

For $K = \mathbb{Q}(\zeta_3)$, $S = \emptyset$, any $p \equiv 1 \pmod{3}$, $p \mid p$, Conjecture 2 holds in depth 1:

$$X(\mathcal{O}_p)_{\emptyset,1} = X(\mathbb{Z}[\zeta_3]).$$

Kim's Conjecture for $\mathbb{Z}[\zeta_3]$ -points

What about Conjecture 1?

A computation using finite polylogarithms shows $\text{Li}_{p-3}(\zeta_6) \neq 0$
(Betts–Kumpitsch–L.)

$$\Rightarrow (\zeta_6, \zeta_6), (\zeta_6^{-1}, \zeta_6^{-1}) \notin X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{\emptyset, \text{PL}, p-3}$$

$$\Rightarrow X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{\emptyset, \text{PL}, p-3} = \{(\zeta_6, \zeta_6^{-1}), (\zeta_6^{-1}, \zeta_6)\}$$

Corollary

For $K = \mathbb{Q}(\zeta_3)$, $S = \emptyset$, any $p \equiv 1 \pmod{3}$, Conjecture 1 holds in depth $p - 3$:

$$X(\mathbb{Z}[\zeta_3] \otimes \mathbb{Z}_p)_{\emptyset, \text{PL}, p-3} = X(\mathbb{Z}[\zeta_3]).$$

Remark: Conjecturally, $\text{Li}_2(\zeta_6) \neq 0$, which would imply that depth 2 suffices.

Theorem (Li-L.)

Let K be an imaginary quadratic field which is not $\mathbb{Q}(\zeta_3)$. Let p be a prime that splits completely in K and satisfies $p \equiv 1 \pmod{3}$. Let $\mathfrak{p} \mid p$ and assume the p -adic period conjecture. Then

$$X(\mathcal{O}_K) = \emptyset \subsetneq \{\zeta_6, \zeta_6^{-1}\} = X(\mathcal{O}_{\mathfrak{p}})_{\emptyset, \text{PL}, \infty},$$

so Conjecture 2 fails for the polylogarithmic quotient.

S_3 -symmetrisation

Corwin–Dan–Cohen observed a similar phenomenon over \mathbb{Q} and proposed “ S_3 -symmetrisation” as a remedy: define

$$X(\mathcal{O}_p)_{S,PL,n}^{S_3}$$

as the maximal S_3 -stable subset of $X(\mathcal{O}_p)_{S,PL,n}$. Note that

$$X(\mathcal{O}_{K,S}) \subseteq X(\mathcal{O}_p)_{S,n} \subseteq X(\mathcal{O}_p)_{S,PL,n}^{S_3} \subseteq X(\mathcal{O}_p)_{S,PL,n}.$$

C–DC conjectured that $X(\mathcal{O}_p)_{S,PL,n}^{S_3} = X(\mathcal{O}_{K,S})$ for $n \gg 0$.

But in the previous theorem, $X(\mathcal{O}_p)_{\emptyset,PL,\infty} = \{\zeta_6, \zeta_6^{-1}\}$ is already S_3 -stable, so even the S_3 -symmetrised polylogarithmic locus is insufficient!

\Rightarrow need to go beyond the polylogarithmic quotient eventually

Theorem (Li–L.)

Let K be a number field containing an imaginary quadratic field and assume $S \supseteq \{l \mid l\}$ for some rational prime l . Let $p \neq l$ be a prime which splits completely in K and assume the p -adic period conjecture. Then, for all $\mathfrak{p} \mid p$, the Chabauty–Kim locus $X(\mathcal{O}_{\mathfrak{p}})_{S, \text{PL}, \infty}$ contains all roots of unity $\neq 1$ in \mathbb{Z}_p .

When p is large, these roots of unity do not belong to $X(\mathcal{O}_{K,S})$, so Conjecture 2 fails for the polylogarithmic quotient.

Variant of polylogarithms:

$$L_n(z) := \sum_{k=0}^{n-1} \frac{B_k}{k!} \log(z)^k \operatorname{Li}_{n-k}(z)$$

Example:

$$L_1(z) = \operatorname{Li}_1(z) = -\log(1-z),$$

$$L_2(z) = \operatorname{Li}_2(z) - \frac{1}{2} \log(z) \operatorname{Li}_1(z),$$

$$L_3(z) = \operatorname{Li}_3(z) - \frac{1}{2} \log(z) \operatorname{Li}_2(z) + \frac{1}{12} \log(z)^2 \operatorname{Li}_1(z)$$

Theorem (Li-L.)

Let K be an imaginary quadratic field, suppose $S = \{1\}$ is Galois-stable. Then $X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{S, \text{PL}, 4}$ is contained in the set of pairs $(z_1, z_2) \in X(\mathbb{Z}_p) \times X(\mathbb{Z}_p)$ satisfying

$$\log(z_1) = \log(z_2),$$

$$\log(1 - z_1) = \log(1 - z_2),$$

$$L_2(z_1) + L_2(z_2) = 0,$$

$$b_{\tau_1} b_{\sigma_2} (L_3(z_1) - L_3(z_2)) = 2b_{\tau_1 \sigma_2} \log(z_1) L_2(z_1),$$

$$b_{\tau_1} b_{\sigma_3} (L_4(z_1) + L_4(z_2)) = b_{\tau_1 \sigma_3} \log(z_1) (L_3(z_1) + L_3(z_2)),$$

for certain constants $b_u \in \mathbb{Q}_p$.

Equations for the depth 1 locus:

$$\log(z_1) = \log(z_2), \quad \log(1 - z_1) = \log(1 - z_2) \quad (*)$$

Lemma

Two types of solutions to (*):

- ▶ *diagonal*: (z, z) , $z \in X(\mathbb{Z}_p)$
- ▶ *off-diagonal*: pairs

$$\left(\frac{1 - \eta}{\zeta - \eta}, \zeta \frac{1 - \eta}{\zeta - \eta} \right)$$

where ζ, η are roots of unity in \mathbb{Z}_p with $\zeta, \eta \neq 1$ and $\zeta \neq \eta$

Sets S of size 1, Galois-stable case

Depth 1 locus $X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{S, \text{PL}, 1}$ is infinite due to diagonal solutions. Additional equation in depth 2:

$$L_2(z_1) + L_2(z_2) = 0$$

Corollary

The depth 2 locus $X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{S, \text{PL}, 2}$ is finite.

Theorem

Kim's Conjecture holds in depth 2 for $K = \mathbb{Q}(i)$, $S = \{(1 - i)\}$, $p = 5$:

$$X(\mathbb{Z}[i] \otimes \mathbb{Z}_5)_{S, \text{PL}, 2} = \left\{ 2, -1, \frac{1}{2}, i, -i, 1 + i, 1 - i, \frac{1+i}{2}, \frac{1-i}{2} \right\}.$$

Theorem (Li-L.)

Let K be an imaginary quadratic field, $S = \{l\}$ not Galois-stable. Then $X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{S, \text{PL}, 2}$ is contained in the set of pairs $(z_1, z_2) \in X(\mathbb{Z}_p) \times X(\mathbb{Z}_p)$ satisfying

$$\begin{aligned}\log^{p^2}(\alpha) \log(z_1) &= \log^{p^1}(\alpha) \log(z_2), \\ \log^{p^2}(\alpha) \log(1 - z_1) &= \log^{p^1}(\alpha) \log(1 - z_2), \\ L_2(z_1) + L_2(z_2) &= 0,\end{aligned}$$

where α is a generator of $\mathcal{O}_{K,S}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and \log^{p_i} is the p_i -adic logarithm on $K_{p_i} \cong \mathbb{Q}_p$.

Theorem (Li-L.)

For $K = \mathbb{Q}(\zeta_8)$, $S = \{(1 - \zeta_8)\}$, the depth-3 locus

$$X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{S, \text{PL}, 3}$$

is contained in the set of quadruples (z_1, z_2, z_3, z_4) satisfying

$$X_1 = X_2, \quad X_3 = X_4,$$

$$Y_1 = Y_2, \quad Y_3 = Y_4,$$

$$Z_1 + Z_2 + Z_3 + Z_4 = 0,$$

$$Z_1 + Z_2 = c_1(-X_3 Y_1 + X_1 Y_3),$$

$$W_1 - W_2 - W_3 + W_4 + c_2(X_2 + X_4)(Z_2 + Z_3) + c_3(X_2 - X_4)(Z_2 + Z_4) = 0,$$

$$W_1 - W_2 + W_3 - W_4 + c_4(X_2 - X_4)(Z_2 + Z_3) + c_5(X_2 + X_4)(Z_2 + Z_4) = 0,$$

for some explicit constants $c_1, \dots, c_5 \in \mathbb{Q}_p$, where

$$X_i := \log(z_i), \quad Y_i := L_1(z_i), \quad Z_i := L_2(z_i), \quad W_i := L_3(z_i).$$

A cyclotomic example

Remarks:

- ▶ Computationally, $\#X(\mathcal{O}_p)_{S,PL,3} = 233$ for $p \mid p = 17$ but only 75 of them are S -integral points
- ▶ We have depth 4 equations but they contain coefficients which we cannot compute

Also have some results for:

- ▶ real quadratic fields
- ▶ refined Chabauty–Kim loci

3. The theory behind

The motivic Selmer scheme

To obtain equations for $X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{S, \Pi^{\text{ét}}}$ we need to compute the scheme-theoretic image of the localisation map

$$\text{Sel}_{S, \Pi^{\text{ét}}}(X) \xrightarrow{\prod_p \text{loc}_p} \prod_{p|\rho} H_f^1(G_p, \Pi^{\text{ét}}).$$

We work with the **motivic Selmer scheme** (Dan-Cohen, Wewers, Corwin, Brown, ...) and identify loc_p with a **cocycle evaluation map**.

The motivic Selmer scheme

- ▶ $\text{MT}(\mathcal{O}_{K,S}, \mathbb{Q})$: category of mixed Tate motives over $\mathcal{O}_{K,S}$
- ▶ Deligne–Goncharov: **motivic fundamental group** $\pi_1^{\text{mot}}(X, b)$ of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ “in $\text{MT}(\mathcal{O}_{K,S}, \mathbb{Q})$ ”
- ▶ fix quotient $\pi_1^{\text{mot}}(X, b) \twoheadrightarrow \Pi$ in $\text{MT}(\mathcal{O}_{K,S}, \mathbb{Q})$
- ▶ **motivic Selmer scheme** $\text{Sel}_{S, \Pi}^{\text{mot}}(X)$: affine \mathbb{Q} -scheme parametrising Π -torsors in $\text{MT}(\mathcal{O}_{K,S}, \mathbb{Q})$
- ▶ comparison theorem (Betts–Kumpitsch–L.): étale realisation induces

$$\text{Sel}_{S, \Pi}^{\text{mot}}(X)_{\mathbb{Q}_p} \cong \text{Sel}_{S, \Pi}^{\text{ét}}(X)$$

The motivic Selmer scheme

- ▶ have a canonical fibre functor $\omega: \text{MT}(\mathcal{O}_{K,S}, \mathbb{Q}) \rightarrow \text{Vect}_{\mathbb{Q}}$:

$$\omega(M) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathbb{Q}(-n), \text{gr}_{2n}^W(M))$$

- ▶ Tannakian fundamental group:

$$G_S^{\text{MT}} = U_S^{\text{MT}} \rtimes \mathbb{G}_m$$

with U_S^{MT} a free pro-unipotent group

- ▶ motivic Selmer scheme is isomorphic to a space of \mathbb{G}_m -equivariant cocycles:

$$\begin{aligned} \text{Sel}_{S,\Pi}^{\text{mot}}(X) &\cong H^1(G_S^{\text{MT}}, \Pi^\omega) \\ &\cong Z^1(U_S^{\text{MT}}, \Pi^\omega)^{\mathbb{G}_m} \\ &\cong Z^1(\text{Lie}(U_S^{\text{MT}}), \text{Lie}(\Pi^\omega))^{\mathbb{G}_m} \end{aligned}$$

The cocycle evaluation map

Now take $\Pi = \Pi_{\text{PL}}$ the polylogarithmic quotient.

U_S^{MT} acts trivially on Π_{PL}^ω , so cocycles = homomorphisms.

Let $\mathfrak{p} \mid p$. Chatzistamatiou–Ünver construct a **p-adic period element**

$$\eta_{\mathfrak{p}} \in U_S^{\text{MT}}(K_{\mathfrak{p}})$$

converting between two functorial splittings of the weight filtration.

Let $\varepsilon_{\mathfrak{p}} := \log(\eta_{\mathfrak{p}})$. We get an evaluation map

$$\text{Sel}_{S, \text{PL}}^{\text{mot}}(X) \cong \text{Hom}(\text{Lie}(U_S^{\text{MT}}), \text{Lie}(\Pi_{\text{PL}}^\omega))^{\mathbb{G}_m} \xrightarrow{\text{ev}_{\varepsilon_{\mathfrak{p}}}} \text{Lie}(\Pi_{\text{PL}}^{\text{dR}})_{K_{\mathfrak{p}}}$$

Proposition (“localisation = evaluation”)

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{Lie}(U_S^{\mathrm{MT}}), \mathrm{Lie}(\Pi_{\mathrm{PL}}^\omega))_{\mathbb{Q}_p}^{\mathbb{G}_m} & \xrightarrow{\mathrm{ev}_{\varepsilon_p}} & \mathrm{Lie}(\Pi_{\mathrm{PL}}^{\mathrm{dR}})_{\mathcal{K}_p} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Sel}_{S, \mathrm{PL}}(X) & \xrightarrow{\mathrm{loc}_p} & H_f^1(\mathbb{G}_p, \Pi_{\mathrm{PL}}^{\mathrm{ét}}) \end{array}$$

Structure of $\text{Lie}(\Pi_{\text{PL}}^\omega)$:

- ▶ $\text{Lie}(\pi_1^\omega(X, b))$ is freely generated by e_0, e_1
- ▶ polylogarithmic quotient: mod out by nested commutators with more than one e_1
- ▶ basis of $\text{Lie}(\Pi_{\text{PL}}^\omega)$: e_0 and $\text{ad}(e_0)^{n-1}e_1$ ($n \geq 1$)

Structure of $\text{Lie}(U_S^{\text{MT}})$:

- ▶ free Lie algebra; get non-canonical generators by lifting basis from the abelianisation
- ▶ in degree $-n$:

$$\text{Lie}(U_S^{\text{MT}})_{-n}^{\text{ab}} \cong \text{Ext}_{\text{MT}(\mathcal{O}_{K,S}, \mathbb{Q})}^1(\mathbb{Q}(0), \mathbb{Q}(n))^\vee \cong K_{2n-1}(\mathcal{O}_{K,S})_{\mathbb{Q}}^\vee$$

- ▶ dimensions $d_n := \dim_{\mathbb{Q}} K_{2n-1}(\mathcal{O}_{K,S})_{\mathbb{Q}}$ known by Borel

Coordinates on the Selmer scheme

For $n \geq 1$, let $\sigma_{n,1}, \dots, \sigma_{n,d_n} \in \text{Lie}(U_S^{\text{MT}})_{-n}$ be lifts of a basis from the abelianisation. Then the $(\sigma_{n,i})_{n,i}$ generate $\text{Lie}(U_S^{\text{MT}})$ freely.

Notation: $\tau_i := \sigma_{1,i}$.

A graded homomorphism $c: \text{Lie}(U_S^{\text{MT}}) \rightarrow \text{Lie}(\Pi_{\text{PL}}^\omega)$ is uniquely determined by

$$c(\tau_i) = x_i(c)e_0 + y_i(c)e_1 \quad (1 \leq i \leq d_1),$$

$$c(\sigma_{n,i}) = z_{n,i}(c) \text{ad}(e_0)^{n-1}e_1 \quad (n \geq 2, 1 \leq i \leq d_n).$$

\Rightarrow Selmer scheme is affine space

$$\text{Sel}_{S,\text{PL}}^{\text{mot}}(X) \cong \text{Spec } \mathbb{Q}[(x_i)_i, (y_i)_i, (z_{n,i})_{n,i}]$$

The cocycle evaluation map in coordinates

Proposition

Each element ε of $\text{Lie}(U_S^{\text{MT}})(R)$ for a \mathbb{Q} -algebra R can uniquely be written in the form

$$\varepsilon = \sum_{n=1}^{\infty} \varepsilon_n + \varepsilon_{\text{Gon}}$$

where ε_{Gon} belongs to an ideal which is mapped to zero under any graded homomorphism to $\text{Lie}(\Pi_{\text{PL}}^{\omega})_R$, $\varepsilon_1 = \sum_{i=1}^{d_1} b_{\tau_i} \tau_i$, and

$$\begin{aligned} \varepsilon_n = & \sum_{i_1 \leq \dots \leq i_{n-1} > i_n} b_{\tau_{i_1} \dots \tau_{i_n}} \text{ad}(\tau_{i_1}) \cdots \text{ad}(\tau_{i_{n-1}}) \tau_{i_n} \\ & + \sum_{m=2}^n \sum_{i=1}^{d_n} \sum_{i_1 \leq \dots \leq i_{n-m}} b_{\tau_{i_1} \dots \tau_{i_{n-m}} \sigma_{m,i}} \text{ad}(\tau_{i_1}) \cdots \text{ad}(\tau_{i_{n-m}}) \sigma_{m,i} \end{aligned}$$

for $n \geq 2$, with coefficients $b_w \in R$, and all indices $i_j \in \{1, \dots, d_1\}$.

The cocycle evaluation map in coordinates

Write the period element $\varepsilon_p \in \text{Lie}(U_S^{\text{MT}})_{K_p}$ in this form, with coefficients $b_w \in K_p$. We get explicit polynomials

$$f_n \in K_p[(x_i)_i, (y_i)_i, (z_{n,i})_{n,i}]$$

whose coefficients are linear combinations of the b_w , describing the cocycle evaluation map

$$\text{Hom}(\text{Lie}(U_S^{\text{MT}}), \text{Lie}(\Pi_{\text{PL}}^\omega))_{\mathbb{Q}_p}^{\mathbb{G}_m} \xrightarrow{\text{ev}_{\varepsilon_p}} \text{Lie}(\Pi_{\text{PL}}^{\text{dR}})_{K_p},$$
$$c \mapsto c(\varepsilon_p) = \sum_{i=1}^{\infty} b_{\tau_i} x_i e_0 + \sum_{i=1}^{\infty} b_{\tau_i} y_i e_1 + \sum_{n=2}^{\infty} f_n(c) \text{ad}(e_0)^{n-1} e_1$$

Determining the coefficients $b_w \in K_p$ in the expansion of ε_p is difficult!

The p -adic period conjecture implies that all b_w are $\neq 0$.

This includes e.g. the p -adic zeta values $\zeta(2n+1) \in \mathbb{Q}_p$, $n \geq 1$.

In degree -1 : assume the basis $\tau_1, \dots, \tau_{d_1}$ is dual to $\alpha_1, \dots, \alpha_{d_1} \in \mathcal{O}_{K,S}^\times \otimes \mathbb{Q}$ under the isomorphism

$$\mathrm{Lie}(U_S^{\mathrm{MT}})_{-1} = \mathrm{Lie}(U_S^{\mathrm{MT}})_{-1}^{\mathrm{ab}} \cong K_1(\mathcal{O}_{K,S})_{\mathbb{Q}}^\vee \cong (\mathcal{O}_{K,S}^\times \otimes \mathbb{Q})^\vee.$$

Then

$$\varepsilon_p = b_{\tau_1} \tau_1 + \dots + b_{\tau_{d_1}} \tau_{d_1} + \dots$$

with

$$b_{\tau_i} = \log^p(\alpha_i) \in K_p.$$

We get equations for the image of

$$\mathrm{Hom}(\mathrm{Lie}(U_S^{\mathrm{MT}}), \mathrm{Lie}(\Pi_{\mathrm{PL}}^\omega))_{\mathbb{Q}_p}^{\mathbb{G}_m} \xrightarrow{\prod_{\mathfrak{p}|p} \mathrm{ev}_{\varepsilon_{\mathfrak{p}}}} \prod_{\mathfrak{p}|p} \mathrm{Lie}(\Pi_{\mathrm{PL}}^{\mathrm{dR}})_{K_{\mathfrak{p}}}$$

by relating the various $\varepsilon_{\mathfrak{p}}$ ($\mathfrak{p} \mid p$) to each other.

Assume that K/\mathbb{Q} is Galois and S is Galois-stable
 $\Rightarrow \mathrm{Gal}(K/\mathbb{Q})$ acts on U_S^{MT} , hence on $\mathrm{Lie}(U_S^{\mathrm{MT}})$

We have

$$\varepsilon_{\sigma(\mathfrak{p})} = \sigma(\varepsilon_{\mathfrak{p}})$$

Choose the generators $\sigma_{n,i}$ in such a way that we know how $\mathrm{Gal}(K/\mathbb{Q})$ acts on them, using

$$\mathrm{Lie}(U_S^{\mathrm{MT}})_{-n}^{\mathrm{ab}} \cong K_{2n-1}(\mathcal{O}_{K,S})_{\mathbb{Q}}^{\vee}.$$

Then the expansion of $\varepsilon_{\sigma(\mathfrak{p})}$ can be obtained from that of $\varepsilon_{\mathfrak{p}}$.

Example: imaginary quadratic fields

Let K be an imaginary quadratic field, $S = \emptyset$.

Let $\sigma \in \text{Gal}(K/\mathbb{Q})$ be complex conjugation.

$\dim_{\mathbb{Q}} K_{2n-1}(K)_{\mathbb{Q}} = 1$ for all $n \geq 2$, whereas $\dim_{\mathbb{Q}} K_{2n-1}(\mathbb{Q})_{\mathbb{Q}} = 0$ for n even.

$\Rightarrow \sigma$ acts as $(-1)^{n+1}$ on $K_{2n-1}(K)_{\mathbb{Q}}$

\Rightarrow can choose generator $\sigma_n \in \text{Lie}(U_S^{\text{MT}})_{-n}$ such that

$$\sigma(\sigma_n) = (-1)^{n+1} \sigma_n$$

Let p be totally split, $p\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$

$$\varepsilon_{\mathfrak{p}_1} = b_2\sigma_2 + b_3\sigma_3 + b_4\sigma_4 + \dots$$

$$\Rightarrow \varepsilon_{\mathfrak{p}_2} = -b_2\sigma_2 + b_3\sigma_3 - b_4\sigma_4 \pm \dots$$

Example: imaginary quadratic fields

Let $c: \text{Lie}(U_S^{\text{MT}}) \rightarrow \text{Lie}(\Pi_{\text{PL}}^\omega)$ be a graded homomorphism. Coordinates $z_n = z_n(c)$ defined by $c(\sigma_n) = z_n \text{ad}(e_0)^{n-1} e_1$.

$$c(\varepsilon_{p_1}) = \sum_{n=2}^{\infty} b_n z_n \text{ad}(e_0)^{n-1} e_1,$$

$$c(\varepsilon_{p_2}) = \sum_{n=2}^{\infty} (-1)^{n+1} b_n z_n \text{ad}(e_0)^{n-1} e_1$$

Let $L_0, L_1, L_2, \dots \in \text{Lie}(\Pi_{\text{PL}}^\omega)^\vee$ be the linear duals of $e_0, e_1, [e_0, e_1], \dots$, and let $L_n^{(i)} := L_n \circ \text{pr}_i$ ($i = 1, 2$).

The image of $\text{ev}_{\varepsilon_{p_1}} \times \text{ev}_{\varepsilon_{p_2}}$ is contained in the subspace of $\text{Lie}(\Pi_{\text{PL}}^\omega) \times \text{Lie}(\Pi_{\text{PL}}^\omega)$ defined by $L_0^{(1)} = L_0^{(2)} = L_1^{(1)} = L_1^{(2)} = 0$ and

$$L_n^{(1)} + (-1)^n L_n^{(2)} = 0$$

Example: imaginary quadratic fields

We obtain the theorem mentioned earlier:

Theorem (Li-L.)

For K imaginary quadratic, $S = \emptyset$, any $1 \leq n \leq \infty$, the locus

$$X(\mathcal{O}_K \otimes \mathbb{Z}_p)_{\emptyset, \text{PL}, n}$$

is contained in the set of pairs $(z_1, z_2) \in X(\mathbb{Z}_p) \times X(\mathbb{Z}_p)$ satisfying

$$\begin{aligned} \log(z_1) &= \log(z_2) = 0, \\ \log(1 - z_1) &= \log(1 - z_2) = 0, \\ \text{Li}_k(z_1) + (-1)^k \text{Li}_k(z_2) &= 0 \text{ for } 2 \leq k \leq n. \end{aligned}$$

Thanks for listening