Rational points and the étale fundamental group

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Overview

1. The étale fundamental group

Topological invariants of schemes

Important invariants for a topological space X:

1. cohomology groups $\mathrm{H}^n(X, A)$, where A is any abelian group;
2. fundamental groups $\pi_n(X, x)$, where $x \in X$ is a base point.

Question

Can we define attach similar invariants to an algebraic variety?
Or even an arbitrary scheme?

One of the great achievements of Grothendieck, Deligne et. al. was the invention of étale cohomology for schemes. It was the key to Deligne's proof of the famous Weil conjectures by transferring topological results like the Lefschetz fixed-point theorem to the world of schemes.

Grothendieck, Artin–Mazur, Friedlander also constructed étale fundamental groups $\pi_n^{\text{ét}}(X, x)$. We focus on $\pi_1^{\text{ét}}(X, x)$.

Étale morphisms

The definition of both étale cohomology and étale fundamental groups rests on the notion of étale morphism:

Definition

A morphism of schemes $f\colon Y \to X$ is étale if it is flat, locally of finite presentation and for every geometric point $x \in X(\Omega)$, with Ω algebraically closed, the fibre $f^{-1}(x) = x \times_X Y$ is a disjoint union of copies of $\operatorname{Spec}(\Omega)$.

Roughly: étale = local homeomorphism

Étale morphisms: examples

Example: $f_n \colon \mathbb{G}_{m,\mathbb{Q}} \to \mathbb{G}_{m,\mathbb{Q}}, y \mapsto y^n$ is étale. The fibre over $x \in \Omega^\times$ is

$$\begin{aligned} f_n^{-1}(x) &= \operatorname{Spec}(\Omega[T]/(T^n - x)) \\ &= \coprod_{\zeta \in \mu_n} \operatorname{Spec}(\Omega[T]/(T - \zeta\sqrt[n]{x})) \\ &\cong \coprod_{\zeta \in \mu_n} \operatorname{Spec}(\Omega). \end{aligned}$$

Non-example: $f_n \colon \mathbb{A}^1_\mathbb{Q} \to \mathbb{A}^1_\mathbb{Q}$, $y \mapsto y^n$ is not étale. The fibre over $0 \in \Omega$ is non-reduced:

$$f_n^{-1}(0) = \operatorname{Spec}(\Omega[T]/(T^n)).$$

Étale morphisms: examples

Example

If L/K is a finite separable field extension, then $\pi\colon \operatorname{Spec}(L) \to \operatorname{Spec}(K)$ is étale.

Proof.

Primitive element theorem: $L = K[T]/(f)$ with f separable. Let $x\colon K \hookrightarrow \Omega$ with Ω algebraically closed. Then $f(T) = (T-\alpha_1)\cdots(T-\alpha_n)$ with $\alpha_i \in \Omega$ pairwise distinct. Then

$$\begin{aligned}\Omega \otimes_K L = \Omega \otimes_K K[T]/(f) = \Omega[T]/(f) \\ \overset{\text{CRT}}{=} \prod_i \Omega[T]/(T-\alpha_i) \cong \prod_i \Omega,\end{aligned}$$

so $\pi^{-1}(x) \cong \coprod_{i=1}^n \operatorname{Spec}(\Omega)$. $\square$

Étale cohomology

Let X be a scheme. An étale covering of X is a family of étale morphisms $(f_i\colon U_i \to X)_{i\in I}$ such that $\bigcup_i f_i(U_i) = X$. This defines a "Grothendieck topology".

An étale sheaf of abelian groups on X is a functor

$$F\colon (\mathrm{Sch}\,/X)^{\mathrm{op}} \to \mathrm{Ab}$$

satisfying the sheaf condition for all étale coverings. The category of étale sheaves on X has enough injectives, so one can define étale cohomology $\mathrm{H}^n_{\text{ét}}(X, F)$ as the right derived functor of the global sections functor $\Gamma(X, F) := F(X)$.

Comparison theorem: For a nonsingular variety $X/\mathbb{C}$, one has $\mathrm{H}^n_{\text{ét}}(X, A) \cong \mathrm{H}^n(X(\mathbb{C}), A)$ for all $n \geq 0$ and finite groups A.

Covering spaces

There are two ways of defining the fundamental group $\pi_1(X, x)$ of a topological space:

- as homotopy classes of loops $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x$;
- via monodromy on fibres of covering spaces as follows:

A covering space of X is a continuous map $f\colon Y \to X$ such that every point in X has a neighbourhood U such that $f^{-1}(U)$ is a disjoint union of copies of U. Let $\mathrm{Cov}(X)$ be the category of covering spaces of X.

Examples:

- $f_n\colon \mathbb{C}^\times \to \mathbb{C}^\times$, $y \mapsto y^n$ is a covering (of degree n);
- $\exp\colon \mathbb{C} \to \mathbb{C}^\times$, $y \mapsto \exp(y)$ is a covering (of infinite degree).

Recovering the fundamental group from $\mathsf{Cov}(X)$

$\pi_1(X,x)$ acts on the fibres $\mathrm{Fib}_x(Y) \coloneqq f^{-1}(x)$ via path-lifting:

Given a loop $\gamma\colon [0,1] \to X$ based at x and given $y \in \mathrm{Fib}_x(Y)$, then γ lifts uniquely to a path $\tilde{\gamma}\colon [0,1] \to Y$ with $\tilde{\gamma}(0) = y$. Define $\gamma.y \coloneqq \tilde{\gamma}(1) \in \mathrm{Fib}_x(Y)$.

The functor $Y \mapsto \mathrm{Fib}_x(Y)$ defines an equivalence of categories

$$\mathsf{Cov}(X) \simeq \pi_1(X,x)\text{-}\mathsf{Set}\,.$$

This implies that $\pi_1(X,x)$ can be recovered from the category $\mathsf{Cov}(X)$ together with the fibre functor $\mathrm{Fib}_x\colon \mathsf{Cov}(X) \to \mathsf{Set}$ as

$$\pi_1(X,x) = \mathrm{Aut}(\mathrm{Fib}_x).$$

That is, giving an element $\gamma \in \pi_1(X,x)$ is equivalent to giving an automorphism of $\mathrm{Fib}_x(Y)$ for every $Y \in \mathsf{Cov}(X)$, naturally in Y.

The étale fundamental group

The definition of $\pi_1(X, x)$ via covering spaces carries over to schemes.

Let X be any connected scheme and $x \in X(\Omega)$ a geometric point. A finite étale cover of X is a morphism $f\colon Y \to X$ which is finite and étale. Let $\mathrm{Cov}(X)$ be the category of finite étale covers of X. Each fibre $\mathrm{Fib}_x(Y) := f^{-1}(x)$ is a finite set, so we get a functor

$$\mathrm{Fib}_x\colon \mathrm{Cov}(X) \to \mathrm{FinSet}\,.$$

Definition

The étale fundamental group $\pi_1^{\text{ét}}(X, x)$ is defined as the automorphism group of the fibre functor:

$$\pi_1^{\text{ét}}(X, x) := \mathrm{Aut}(\mathrm{Fib}_x).$$

The étale fundamental group: Examples

$\pi_1^{\text{ét}}(X,x)$ is naturally a profinite group.

One gets a tautological equivalence of categories

$$\mathrm{Cov}(X) \simeq \pi_1^{\text{ét}}(X,x)\text{-FinSet}\,.$$

Examples:

- **Complex varieties:** If $X/\mathbb{C}$ is a nonsingular variety, there is an equivalence

  $$\mathrm{Cov}(X) \simeq (\text{finite covers of } X(\mathbb{C})),$$

  which implies

  $$\begin{aligned}\pi_1^{\text{ét}}(X,x) &= \text{profinite completion of } \pi_1(X(\mathbb{C}),x)\\ &= \text{inverse limit of finite quotients of } \pi_1(X(\mathbb{C}),x).\end{aligned}$$

  Note that the comparison fails for infinite covers:
  $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is not algebraic!

The étale fundamental group: Examples

- **Fields:** If K is a field, $K \hookrightarrow \overline{K}$ a separable closure, then the connected finite étale covers of $\mathrm{Spec}(K)$ are $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ with L/K finite separable. This implies

$$\pi_1^{\text{ét}}(\mathrm{Spec}(K), \mathrm{Spec}(\overline{K})) = \mathrm{Gal}(\overline{K}/K).$$

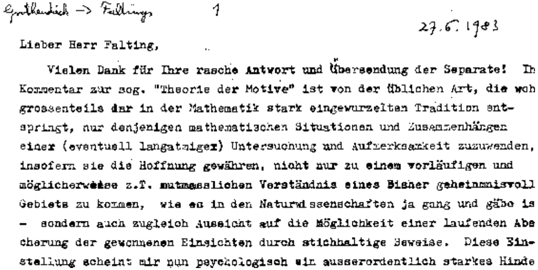
  In other words, the étale fundamental group of $\mathrm{Spec}(K)$ is the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$.
- **Abelian varieties:** For an abelian variety $A/\overline{\mathbb{Q}}$, the étale fundamental group is the Tate module

$$\pi_1^{\text{ét}}(A, 0) = \varprojlim_n A(\overline{\mathbb{Q}})[n] =: T(A).$$

2. The Section Conjecture

Grothendieck's letter to Faltings

In a letter to Faltings from 1983, Grothendieck lays out his vision of what he calls anabelian geometry.


Grothendieck -> Faltings 1

27.6.1983

Lieber Herr Falting,

Vielen Dank für Ihre rasche Antwort und Übersendung der Separate! Ih
Kommentar zur sog. "Theorie der Motive" ist von der üblichen Art, die woh
grossenteils der in der Mathematik stark eingewurzelten Tradition ent-
springt, nur denjenigen mathematischen Situationen und Zusammenhängen
einer (eventuell langatmiger) Untersuchung und Aufmerksamkeit zuzuwenden,
insofern sie die Hoffnung gewähren, nicht nur zu einem vorläufigen und
möglicherweise z.T. mutmasslichen Verständnis eines Bisher geheimnisvoll
Gebiets zu kommen, wie es in den Naturwissenschaften ja gang und gäbe is
– sondern auch zugleich Aussicht auf die Möglichkeit einer laufenden Abs
cherung der gewonnenen Einsichten durch stichhaltige Beweise. Diese Ein-
stellung scheint mir nun psychologisch ein ausserordentlich starkes Hinde

Idea: for a large class of schemes, a lot of information can be recovered from the étale fundamental group.

Example: number fields are completely determined by their absolute Galois group (Neukirch–Uchida theorem):

$$G_K \cong G_L \Rightarrow K \cong L.$$

Anabelian schemes

This class of "anabelian schemes" should include at least the following:

1. finitely generated fields over $\mathbb{Q}$ (Neukirch–Uchida, Pop)
2. hyperbolic curves over such fields (Mochizuki)
3. the moduli stacks $\mathcal{M}_{g,n}$
4. successive fibrations by hyperbolic curves

Fundamental exact sequence

Let X/K be a smooth projective curve of genus ≥ 2 over a number field. Such curves are hyperbolic.

Grothendieck's section conjecture predicts a way of recovering the set of rational points $X(K)$ from the étale fundamental group.

The maps

$$X_{\overline{K}} \to X \to \mathrm{Spec}(K)$$

induce an exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\overline{K}}) \to \pi_1^{\text{ét}}(X) \to G_K \to 1 \qquad \text{(FES)}$$

of étale fundamental groups: the fundamental exact sequence.

If $x \in X(K)$ is a rational point, it induces a section s_x of (FES), well-defined up to $\pi_1^{\text{ét}}(X_{\overline{K}})$-conjugation. So we have a section map

$$X(K) \to \mathrm{Sec}(X/K) := \{\text{sections of (FES)}\}/\sim \,. \qquad \text{(S)}$$

The Section Conjecture

Section Conjecture (Grothendieck, 1983)

If X/K is a smooth projective curve of genus ≥ 2 over a number field, then (S) is a bijection $X(K) \cong \mathrm{Sec}(X/K)$.

- (S) is known to be injective. The question is whether it is surjective.
- Some examples of X have been constructed where one can show that $\mathrm{Sec}(X/K) = \emptyset$ (Stix, Li–Litt–Salter–Srinivasan). In these cases, (S) is bijective automatically.

Open Question

Can we find some X with $X(K) \neq \emptyset$ for which the Section Conjecture holds?

Selmer sections

If s is a section of the fundamental exact sequence and v is a place of K, then the restriction $s|_{G_v}$ is a section of the local FES

$$1 \to \pi_1^{\text{ét}}(X_{\overline{K}_v}) \to \pi_1^{\text{ét}}(X_{K_v}) \to G_v \to 1\,.$$

Definition

We say that s is Selmer (or sometimes locally geometric) if $s|_{G_v}$ comes from a K_v-rational point $x_v \in X(K_v)$ for all places v. We write $\mathrm{Sec}(X/K)^{\mathrm{Sel}}$ for the set of Selmer sections.

So we have

$$X(K) \subseteq \mathrm{Sec}(X/K)^{\mathrm{Sel}} \subseteq \mathrm{Sec}(X/K)\,.$$

Selmer Section Conjecture

If X/K is a smooth projective curve of genus ≥ 2 over a number field, then

$$X(K) = \mathrm{Sec}(X/K)^{\mathrm{Sel}}.$$

Recent joint work with Alex Betts and Theresa Kumpitsch:[1]

> *One can use the Chabauty–Kim method to prove instances of the Selmer Section Conjecture.*

[1] Chabauty–Kim and the Section Conjecture for locally geometric sections arXiv:2305.09462

3. The Chabauty–Kim method

The Chabauty–Kim method

Let $X/\mathbb{Q}$ be a smooth projective curve of genus ≥ 2.

- We know that $X(\mathbb{Q})$ is finite by Faltings' Theorem.
- It is very difficult to provably compute $X(\mathbb{Q})$ in general.
- The Chabauty–Coleman method can sometimes be used, but it requires $r < g$.
- Minhyong Kim (2005) developed a non-abelian generalisation which can be used for more general curves, called Chabauty–Kim method or non-abelian Chabauty.
- It is still largely conjectural but has seen some spectacular successes in recent years.

The Chabauty–Kim diagram

Assume $b \in X(\mathbb{Q})$. Let p be an auxiliary prime.

The Chabauty–Kim method works with the $\mathbb{Q}_p$-pro-unipotent completion U of the étale fundamental group $\Pi := \pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}, b)$.

It comes with a continuous homomorphism

$$\phi \colon \Pi \to U(\mathbb{Q}_p)$$

and with an action of $G_{\mathbb{Q}}$.

We have the Chabauty–Kim diagram

$$\begin{array}{ccc} X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) \\ \downarrow j & & \downarrow j_p \\ \mathrm{Sel}_U(X/\mathbb{Q})(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, U)(\mathbb{Q}_p) \end{array}$$

where $\mathrm{Sel}_U(X/\mathbb{Q})$ is called the global Selmer scheme.

More generally, U can be replaced with a $G_{\mathbb{Q}}$-equivariant quotient.

Finiteness via Chabauty–Kim

$$\begin{array}{ccc} X(\mathbb{Q}) & \hookrightarrow & X(\mathbb{Q}_p) \\ \downarrow j & & \downarrow j_p \\ \mathrm{Sel}_U(X/\mathbb{Q})(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, U)(\mathbb{Q}_p) \end{array}$$

Fact: loc_p is an algebraic map of affine $\mathbb{Q}_p$-schemes

Conjecture

loc_p *is not dominant for sufficiently large* U*.*

If satisfied (e.g., for dimension reasons), we get finiteness of $X(\mathbb{Q})$.

Proof:

- Let $0 \neq f\colon \mathrm{H}^1_f(G_p, U) \to \mathbb{A}^1$ vanishing on $\mathrm{im}(\mathrm{loc}_p)$
- the pullback $f \circ j_p\colon X(\mathbb{Q}_p) \to \mathbb{Q}_p$ is a nonzero p-adic analytic function whose vanishing set is finite and contains $X(\mathbb{Q})$

Kim's Conjecture

Definition

The Chabauty–Kim locus associated to U is the set

$$X(\mathbb{Q}_p)_U := \{x \in X(\mathbb{Q}_p) \ : \ j_p(x) \in \mathrm{im}(\mathrm{loc}_p)\} \subseteq X(\mathbb{Q}_p)\,.$$

Commutativity of the Chabauty–Kim diagram gives

$$X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)_U \subseteq X(\mathbb{Q}_p)\,.$$

In the particular case that U is the whole fundamental group, we write $X(\mathbb{Q}_p)_\infty$ for $X(\mathbb{Q}_p)_U$. This is contained in all other $X(\mathbb{Q}_p)_U$'s.

Kim's Conjecture

$$X(\mathbb{Q}_p)_\infty = X(\mathbb{Q}).$$

4. Our Results

Kim's conjecture implies Selmer Section Conjecture

We make precise the relationship between Kim's Conjecture and the Selmer Section Conjecture.

Theorem A (Betts–Kumpitsch–L.), projective case

Let $X/\mathbb{Q}$ be a smooth projective curve of genus ≥ 2 with $X(\mathbb{Q}) \neq \emptyset$. Suppose that Kim's Conjecture holds for (X, p) for p in a set $\mathfrak{P}$ of primes of Dirichlet density 1. Then the Selmer Section Conjecture holds for X.

This gives a new strategy for proving instances of the Selmer Section Conjecture.

We show that the strategy is viable by verifying the hypotheses in an example of an *affine* hyperbolic curve, the thrice-punctured line over $\mathbb{Z}[1/2]$.

Generalisation: S-integral points

Now fix a finite set S of primes. Let $Y/\mathbb{Q}$ be a hyperbolic curve, and let $\mathcal{Y}/\mathbb{Z}_S$ be an S-integral model of Y.

Definition

A section s of the fundamental exact sequence for Y is S-Selmer (with respect to the model $\mathcal{Y}$) if $s|_{G_p}$ comes from a

$$\begin{cases} \mathbb{Q}_p\text{-rational point } y_p \in Y(\mathbb{Q}_p) & \text{if } p \in S, \\ \mathbb{Z}_p\text{-integral point } y_p \in \mathcal{Y}(\mathbb{Z}_p) & \text{if } p \notin S, \end{cases}$$

for all primes p. We write $\mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}}$ for the set of S-Selmer sections.

So we have

$$\mathcal{Y}(\mathbb{Z}_S) \subseteq \mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}} \subseteq \mathrm{Sec}(Y/\mathbb{Q})\,.$$

Conjecture (S-Selmer Section Conjecture)

$$\mathrm{Sec}(\mathcal{Y}/\mathbb{Z}_S)^{\mathrm{Sel}} = \mathcal{Y}(\mathbb{Z}_S)\,.$$

Chabauty–Kim for S-integral points

There is also a version of the Chabauty–Kim method which applies to S-integral points on $\mathcal{Y}$, thus defining Chabauty–Kim loci

$$\mathcal{Y}(\mathbb{Z}_S) \subseteq \mathcal{Y}(\mathbb{Z}_p)_{S,U} \subseteq \mathcal{Y}(\mathbb{Z}_p).$$

Kim's Conjecture

$$\mathcal{Y}(\mathbb{Z}_p)_{S,\infty} = \mathcal{Y}(\mathbb{Z}_S).$$

Theorem A (Betts–Kumpitsch–L.)

Let $\mathcal{Y}/\mathbb{Z}_S$ be a hyperbolic curve whose smooth completion has a $\mathbb{Q}$-rational point. Suppose that Kim's Conjecture holds for $(\mathcal{Y}, S, p)$ for p in a set $\mathfrak{P}$ of primes of Dirichlet density 1. Then the S-Selmer Section Conjecture holds for $(\mathcal{Y}, S)$.

Theorem B

We can verify the hypotheses of Theorem A in one example.

Theorem B (Betts–Kumpitsch–L.)

Let $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$ be the thrice-punctured line over $\mathbb{Z}[1/2]$. Then Kim's Conjecture holds for $(\mathcal{Y}, \{2\}, p)$ for all odd primes p.

Consequence: The S-Selmer Section Conjecture holds for $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$ and $S = \{2\}$.

5. Proof of Theorem A

Theorem A, recap

Theorem A, projective case

Let $X/\mathbb{Q}$ be a smooth projective curve of genus ≥ 2 with $X(\mathbb{Q}) \neq \emptyset$. Suppose that Kim's Conjecture holds for (X, p) for p in a set $\mathfrak{P}$ of primes of Dirichlet density 1. Then the Selmer Section Conjecture holds for X.

Let $s \in \mathrm{Sec}(X/\mathbb{Q})^{\mathrm{Sel}}$ be a Selmer section, so $s|_{G_p}$ is induced by some $x_p \in X(\mathbb{Q}_p)$ for all primes p.

Claim

$x_p \in X(\mathbb{Q}_p)_\infty$ for all primes p.

Proof: Let $\Pi = \pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}, b)$ be the profinite étale fundamental group based at some $b \in X(\mathbb{Q})$. The section map

$$X(\mathbb{Q}) \to \mathrm{Sec}(X/\mathbb{Q})$$

can be identified with the map

$$X(\mathbb{Q}) \to \mathrm{H}^1(G_\mathbb{Q}, \Pi)$$

sending a point $x \in X(\mathbb{Q})$ to the cocycle $c\colon G_\mathbb{Q} \to \Pi$ measuring the difference between the two sections s_b and s_x:

$$c(\sigma) = s_x(\sigma) s_b(\sigma)^{-1}.$$

Have a similar local section map $X(\mathbb{Q}_\ell) \to \mathrm{H}^1(G_\ell, \Pi)$.

Consider the $\mathbb{Q}_p$-pro-unipotent completion map $\phi\colon \Pi \to U(\mathbb{Q}_p)$. For any prime ℓ, we then have a commuting diagram

$$\begin{array}{ccc} & & X(\mathbb{Q}_\ell) \\ & & \downarrow \\ s \in \mathrm{H}^1(G_{\mathbb{Q}}, \Pi) & \xrightarrow{\mathrm{loc}_\ell} & \mathrm{H}^1(G_\ell, \Pi) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{H}^1(G_{\mathbb{Q}}, U(\mathbb{Q}_p)) & \xrightarrow{\mathrm{loc}_\ell} & \mathrm{H}^1(G_\ell, U(\mathbb{Q}_p)) \end{array}$$

Selmer sections: elements of $\mathrm{H}^1(G_{\mathbb{Q}}, \Pi)$ locally coming from $X(\mathbb{Q}_\ell)$
Selmer scheme: elements of $\mathrm{H}^1(G_{\mathbb{Q}}, U(\mathbb{Q}_p))$ locally coming from $X(\mathbb{Q}_\ell)$

Since s is Selmer, $\phi_*(s) \in \mathrm{H}^1(G_{\mathbb{Q}}, U(\mathbb{Q}_p))$ lies in the Selmer scheme $\mathrm{Sel}_\infty(X/\mathbb{Q})$.

Take $\ell = p$ in the above diagram and restrict to Selmer elements:

$$\begin{array}{ccc}
 & & x_p \in X(\mathbb{Q}_p) \\
 & & \Big\downarrow{\scriptstyle j_p} \\
s \in \mathrm{Sec}(X/\mathbb{Q})^{\mathrm{Sel}} & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1(G_p, \Pi) \\
\Big\downarrow{\scriptstyle \phi_*} & & \Big\downarrow{\scriptstyle \phi_*} \\
\mathrm{Sel}_\infty(X/\mathbb{Q})(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1(G_p, U(\mathbb{Q}_p))
\end{array}$$

Hence

$$j_p(x_p) = \mathrm{loc}_p(\phi_*(s)) \in \mathrm{loc}_p(\mathrm{Sel}_\infty(X/\mathbb{Q})),$$

and so $x_p \in X(\mathbb{Q}_p)_\infty$ as claimed. □

If Kim's Conjecture holds for (X, p) for all $p \in \mathfrak{P}$, then the preceding claim shows that each

$$x_p \in X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)$$

is one of the finitely many rational points on X.

We proceed by proving:

1. The rational points x_p for $p \in \mathfrak{P}$ all agree, call it x.
2. The original Selmer section s is the one induced by x.

This shows that X satisfies the Selmer Section Conjecture.

6. Proof of Theorem B

Theorem B, recap

Theorem B

Let $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}[1/2]} \smallsetminus \{0, 1, \infty\}$ be the thrice-punctured line over $\mathbb{Z}[1/2]$. Then Kim's Conjecture holds for $(\mathcal{Y}, \{2\}, p)$ for all odd primes p.

Recall that Kim's conjecture says that the inclusion

$$\mathcal{Y}(\mathbb{Z}[1/2]) \subseteq \mathcal{Y}(\mathbb{Z}_p)_{\{2\},\infty}$$

is an equality.

Note that $\mathbb{Z}[1/2]^\times = \{\pm 2^n : n \in \mathbb{Z}\}$ and

$$\mathcal{Y}(\mathbb{Z}[1/2]) = \{z \in \mathbb{Z}[1/2]^\times \text{ s.t. } 1 - z \in \mathbb{Z}[1/2]^\times\} = \{2, -1, \tfrac{1}{2}\}.$$

Key ideas of the proof

- The Selmer scheme has 3 irreducible components, one for each cusp:

$$\mathrm{Sel}_\infty(\mathcal{Y}/\mathbb{Z}[1/2]) = \mathrm{Sel}_\infty(\mathcal{Y}/\mathbb{Z}[1/2])^0 \cup \mathrm{Sel}_\infty(\mathcal{Y}/\mathbb{Z}[1/2])^1 \cup \mathrm{Sel}_\infty(\mathcal{Y}/\mathbb{Z}[1/2])^\infty.$$

  Accordingly, the Chabauty–Kim locus is a union

$$\mathcal{Y}(\mathbb{Z}_p)_{\{2\},\infty} = \mathcal{Y}(\mathbb{Z}_p)^0_{\{2\},\infty} \cup \mathcal{Y}(\mathbb{Z}_p)^1_{\{2\},\infty} \cup \mathcal{Y}(\mathbb{Z}_p)^\infty_{\{2\},\infty}.$$

  This corresponds to the partition of

$$\mathcal{Y}(\mathbb{Z}[1/2]) = \{2\} \cup \{-1\} \cup \{\tfrac{1}{2}\}$$

  into mod-2 reduction types.
- Consider only one component and exploit symmetries.
- Use the polylogarithmic quotient U_{PL} of the fundamental group and prove a motivic–étale comparison theorem for Selmer schemes to import results by Corwin–Dan-Cohen.

Explicit localisation map

Corwin–Dan-Cohen: The localisation map in the Chabauty–Kim diagram

$$\mathrm{loc}_p\colon \mathrm{Sel}_{\mathrm{PL}}(\mathcal{Y}/\mathbb{Z}[1/2])^1 \to \mathrm{H}^1_f(G_p, U_{\mathrm{PL}})$$

is given by

$$\mathrm{loc}_p\colon \operatorname{Spec}\mathbb{Q}_p[y, z_3, z_5, z_7, \ldots] \to \operatorname{Spec}\mathbb{Q}_p[\log, \mathrm{Li}_1, \mathrm{Li}_2, \mathrm{Li}_3, \ldots],$$

$$\begin{aligned}
\mathrm{loc}_p^\sharp \log &= 0,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_1 &= \log(2)y,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_2 &= 0,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_3 &= \zeta(3)z_3,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_4 &= 0,\\
\mathrm{loc}_p^\sharp \mathrm{Li}_5 &= \zeta(5)z_5,\\
&\vdots
\end{aligned}$$

Coleman functions vanishing on the Chabauty–Kim locus

We find infinitely many functions on $\mathrm{H}^1_f(G_p, U_{\mathrm{PL}})$ which vanish on the image of the Selmer scheme. Pulling back the functions along j_p shows:

Proposition

The following functions vanish on the Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)^1_{\{2\},\infty}$:

$$\log(z) \quad \textit{and} \quad \mathrm{Li}_k(z) \textit{ for } k \geq 2 \textit{ even.}$$

Here, log and Li_k are p-adic analytic functions on $\mathcal{Y}(\mathbb{Z}_p)$ defined as iterated Coleman integrals:

$$\log(z) = \int_0^z \frac{\mathrm{d}z}{z}, \quad \mathrm{Li}_k(z) = \int_0^z \underbrace{\frac{\mathrm{d}z}{z} \cdots \frac{\mathrm{d}z}{z} \frac{\mathrm{d}z}{1-z}}_{k}.$$

Proof of Kim's conjecture

Now Kim's conjecture for $\mathcal{Y}/\mathbb{Z}[1/2]$ follows from:

Proposition

The only common zero in $\mathcal{Y}(\mathbb{Z}_p)$ of the functions $\log(z)$ and $\mathrm{Li}_k(z)$ for $k \geq 2$ even is $z = -1$.

$\log(z) = 0$ implies that z is a $(p-1)$-st root of unity in $\mathbb{Z}_p$.

Li_k has a mod-p variant $\mathrm{li}_k \colon \mathbb{F}_p \to \mathbb{F}_p$ given by

$$\mathrm{li}_k(z) = \sum_{i=1}^{p-1} \frac{z^i}{i^k}.$$

$\mathrm{Li}_k(z) = 0$ implies $\mathrm{li}_k(\bar{z}) = 0$. Take $k = p-3$ and use little Fermat:

$$\mathrm{li}_{p-3}(z) = \sum_{i=1}^{p-1} i^{-(p-3)} z^i = \sum_{i=1}^{p-1} i^2 z^i = z(z+1)(z-1)^{p-3}.$$

Proof of Kim's conjecture

At this point we have:

(1) $z \in \mathcal{Y}(\mathbb{Z}_p)$

(2) z is a $(p-1)$-st root of unity

(3) $\overline{z} := z \bmod p$ is a zero of $\mathrm{li}_{p-3}(z) = z(z+1)(z-1)^{p-3}$

(3) implies $\overline{z} \in \{0, -1, 1\}$.

But by (1), $\overline{z}$ is in $\mathcal{Y}(\mathbb{F}_p) = \mathbb{F}_p \smallsetminus \{0, 1\}$, so $\overline{z} = -1$.

Finally, (2) implies $z = -1$.

This shows that

$$\{-1\} \subseteq \mathcal{Y}(\mathbb{Z}_p)^1_{\{2\},\infty}$$

is an equality, hence Kim's conjecture holds.

7. Further directions

Further directions

- In ongoing work with David Corwin and Ishai Dan-Cohen, we are extending the Chabauty–Kim method for $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ beyond the polylogarithmic quotient. The resulting functions for the CK locus involve multiple polylogarithms like $\mathrm{Li}_{3,1}$ and $\mathrm{Li}_{2,1,1}$.
- As a first step towards higher dimensions, Ishai Dan-Cohen and David Jarossay have applied the Chabauty–Kim method to the surface $\mathcal{M}_{0,5}$ over $\mathbb{Z}[1/6]$ and produced an interesting function for the CK locus.

Thank you for listening